RISK AND COMPOUND RETURN

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S U M M A R Y

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The geometric mean or compound return is (generally) the appropriate measure of multi-period return. Given intertemporally independent and identically distributed returns over time, then we will show that: a) compound return is asymptotically normal, N-period terminal wealth is asymptotically lognormal; b) median N-period terminal wealth is asymptotically a direct function of expected compound return; c) expected compound return decreases as a function of the number of periods.

A statistically tractable approximation of the geometric mean for finite N was used, together with the Ross (1973) model for single-period returns, which showed that: a) expected compound return is approximately a quadratic function of beta; b) a critical beta (generally) exists beyond which expected compound return decreases.

By assuming that the distribution of compound return can be approximated by a normal distribution for finite N, risk policies were found which maximize: a) the probability that a given compound return rate or level of N-period terminal wealth will be achieved over the N-period investment horizon; b) the compound return rate or level of N-period terminal wealth that can be achieved in an N-period investment horizon for a specified probability level.

The objective of the analysis has been to clarify the long term risk-return relationship for rational portfolio decision making and to develop new tools for analysing the consequences of a long term investment policy. A risk-averse expected utility maximizer of N-period terminal wealth is thus supplied with a clearer basis on which to determine an appropriate investment policy.

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1. Introduction.

One of the most fundamental concepts in the theory of finance is the assumption that expected return increases with increasing risk, when risk is appropriately defined. Justification for this view can be found in capital asset pricing theory (Sharpe, 1964; Lintner, 1965; Fama, 1968, 1973). For a single-period expected utility of terminal wealth maximizer who chooses among alternative portfolios on the basis of the mean and variance of return, given capital market equilibrium and other assumptions[1], it can be shown that the expected one period return from a security or portfolio is a linear function of its systematic or market risk, β, and is given by

$$\mu = E(R) = R_o + \beta\,(E(M) - R_o) \qquad (1)$$

where

$$\beta = COV(R,M)/\sigma_M^2. \qquad (2)$$

The symbols in (1) and (2) are defined as follows: R is the total return on the capital asset or portfolio for the period; M is the return on the market portfolio (value weighted) of all assets taken together; R_o is the return on a riskless asset for the period; COV(R,M) is the covariance between the asset or portfolio and the market portfolio; σ_M^2 is the variance of return of the market portfolio. Equation (1) is often called the "security market line."

Using (2), and by definition of the correlation, ρ, the standard deviation of returns can be written as:

$$\sigma_R = \beta\sigma_M/\rho. \qquad (3)$$

[1]For a detailed summary, including assumptions, see Sharpe (1970).

Equations (1) and (3) completely specify the equilibrium return-risk relationships for an asset or portfolio of assets for a single investment period under the assumptions of the capital market model.

Fama (1970) provides an analysis of the multi-period consumption-investment problem and a justification for the single-period expected utility of terminal wealth model leading to (1) and (3). However, one of the assumptions of the capital asset pricing model--single-period capital market equilibrium is problematic in a multi-period context. It implies either that intertemporal returns are not identically distributed or that the equilibrium market is equally weighted (Rosenberg and Ohlson, 1973). Further, we will show that, if single-period returns are generated according to (1) and (3), and if a multi-period risk averse investor uses the mean and variance of the geometric mean as a criterion for portfolio selection (which we will attempt to justify), then there is (for many cases of interest) a maximum level of beta beyond which expected geometric mean return decreases while the variance increases. This result implies decreased demand for assets or portfolios with betas greater than the expected geometric mean maximum value.

Ross' (1973) one-factor arbitrage model is an alternative single-period model which leads to the return-risk relationships (1) and (3) without the equilibrium market assumption. It is also consistent with the Fama (1970) analysis. In Ross' model, return for every security in the period is assumed to be generated by a simple linear relationship which is a function of a random variable with zero expectation common to all securities and a zero mean noise random variable. The rate of return R_o is interpreted, under the no arbitrage condition, as the rate of return common to riskless assets. M refers to the rate of return of the market portfolio in the period under the assumption that the market portfolio is "well diversified" with respect to the posited linear return generation process. Under these assumptions, equations (1), (2) and (3) can be used to describe the one-period expected total return, systematic level of risk, and standard deviation of return, respectively, for any security or portfolio.

We will assume the Fama (1970) conditions: multi-period investors are risk averse maximizers of expected utility of consumption and terminal wealth where future consumption and investment opportunities are state independent. These assumptions lead to the result that the behavior of multi-period investors in each period is indistinguishable from a single-period maximizer of expected utility of terminal wealth. Given the single-period investment policy portfolio decisions, β and ρ, our objective will be to examine the intertemporal consequences of returns generated according to the single-period return-risk relationships (1) and (3) for rebalanced portfolios. This will be accomplished by using the mean and variance of the geometric mean or compound return as a measure of multi-period portfolio performance.

The multi-period portfolio compound return model which will be derived is related to the capital growth or geometric mean models of Latane (1959), Markowitz (1959, Ch. 6), Hakansson (1971), and Wippern (1971). Merton and Samuelson (1974) and Fama and MacBeth (1974) provide a general review of this body of work.

Our model differs in two significant respects from previous work: 1) The geometric mean return formula (13), which is the basis of the mean-variance compound return portfolio selection model, is an approximation of N-period compound return for finite N, and is based on the mean and variance of single-period returns so that consistency with the single-period mean-variance maximization of expected utility of terminal wealth assumption is maintained. 2) Although portfolio policies which maximize the expected geometric mean are of interest, the geometric mean is not used as a surrogate utility function; attention is focused primarily on the N-period consequences of following any given portfolio investment policy.

In our investigation of the multi-period portfolio problem, we shall adopt a point of view which has been termed the "Markowitz procedure" by Mossin (1973, p.44). The portfolio selection procedure is broken down into two steps. The first step involves an elimination of certain portfolios in order to form an "efficient" set of portfolios. In the next step, a final selection is made among the portfolios in the efficient set. Our analysis will focus on this first step in the portfolio selection process.

The results have direct application to Monte Carlo portfolio simulation studies which use the market line model for generating portfolio return in each period of an investment horizon (see e.g. Lorie and Hamilton, 1973, Ch. 15). The portfolio simulation technique is typically used when the interrelationships of various factors and assumptions on portfolio return are not well understood. Without an hypothesis of likely behavior, designing simulation experiments and analysing simulation output may be very difficult.

From our analysis we will be able to show the impact of capital market and investment policy parameter assumptions on short and long term portfolio return via the simulation process. This can serve as a benchmark for evaluating how factors specific to a given situation affect investment return and consequent funding behavior. The net effect is to allow the use of the portfolio simulation technique in a more active decision making role. This multi-period model of portfolio return should also prove useful in providing a framework for setting investment policy for pension funds, endowment funds, and other situations in which long range investment goals are of critical importance.

In Section 2 we derive some general results concerning compound return and N-period terminal wealth and provide the motivation for the mean-variance compound return portfolio selection model. Using only the assumption that single-period returns are intertemporally independent and identically distributed, we will show that: a) expected compound return is a decreasing function of the number of periods; b) expected compound return is asymptotically directly related to the median of the N-period terminal wealth distribution; c) the compound return distribution is asymptotically normal.

In Section 3 we state the assumptions and derive the central mathematical results of the N-period mean-variance compound return model. This includes the multi-period geometric mean return security market "line", the risk policy (β) which maximizes expected compound return for a given investment horizon, and a formula for the maximum long-term growth rate (with probability one) available in a given capital market. We also discuss the relationship of the N-period mean-variance compound return efficient portfolios with the single-period mean-variance efficient set.

In Section 4 we display the multi-period security market line for various capital market parameters, analyze the N-period consequences of constant portfolio risk and diversification strategies, and suggest some implications for portfolio management.

In Section 5 we illustrate the relationship between risk policy and compound return probability assuming normality and solve for the risk level which maximizes the probability that a given level of wealth or compound return will be achieved. The risk level which maximizes compound return for a specified probability is also derived. In Section 6 we provide a summary of our results.

2. Compound return and N-period terminal wealth: Some general results.

From simple examples (e.g., Francis and Archer, 1971, p. 13; Latane and Young, 1971, p.978), it is easy to show that (generally) the appropriate measure of portfolio return over an N-period investment horizon (assuming N greater than one) is the distribution of the geometric mean.

The definition of the geometric mean or compound return over N investment periods is:

$$G_N(R) = \sqrt[N]{(1 + r_1)\ (1 + r_2)\dots(1 + r_N)} - 1 \qquad (4)$$

where R will now denote a vector of returns $r_1, r_2, \dots, r_N$ in the N investment periods. We make the assumption that returns r_i, i=1, ..., N, are intertemporally independent and identically distributed, the latter assumption being a mathematical convenience.

Expected compound return is a decreasing function of the number of periods[2]. To show this we write expected compound return in the form

$$E(G_N(R)) = (E(1+r)^{\frac{1}{N}})^N - 1. \qquad (5)$$

$E(G_N(R))$ in (5) defines an L_p norm and can be shown to be a monotone non-increasing (generally decreasing) function of N (Thomas, 1971, p.317).

[2] This result provides an alternative derivation and a reinterpretation of the downward bias properties of the geometric mean found in Blume (1974).

From (5) we can prove, by use of binomial series expansions and other analytic techniques, that:

$$\lim_{N\to\infty} E(G_N(R)) = e^{E(\ln(1+r))}-1 \qquad (6)$$

which is the formula used by Hakansson (1971) for determining long term expected compound return.

We define (assuming existence) the parameters

$$A=E(\ln(1+r))$$
$$B=V(\ln(1+r)). \qquad (7)$$

Then, using a result in Rao (1965, p. 320), we can show that $G_N(R)$ is asymptotically normally distributed with mean $e^A -1$ and variance $e^{2A}B/N$. Therefore, the mean and variance of compound return asymptotically characterize the compound return distribution, and hence are asymptotically the appropriate descriptive parameters of the multi-period portfolio return distribution. As a corollary to this result, the median and mean of compound return asymptotically coincide.

N-period terminal wealth, in units of initial wealth, can be written as:

$$W_N(R)=(1+r_1)(1+r_2)\ \dots\ (1+r_N). \qquad (8)$$

Consequently N-period terminal wealth and the geometric mean are related according to

$$W_N(R) = (G_N(R) + 1)^N. \qquad (9)$$

Let W_p and G_p denote, respectively, the pth quantiles in the terminal wealth and compound return distributions for some fixed investment horizon. Then the fundamental relationship between the probability distribution of N-period terminal wealth and compound return is expressed by:

$$W_p = (G_p + 1)^N. \tag{10}$$

This relationship between the quantiles can be derived from the fact that W_N in (9) is a monotone increasing function of G_N. From (10) it follows that there is a direct relationship between the medians of the two distributions for any N. More generally, the mean and variance of compound return provide, via (10), an asymptotic description of the N-period terminal wealth distribution.

The asymptotic distribution of the N-period terminal wealth distribution is lognormal. This can easily be shown by applying the central limit theorem to the log of (8).

It should be noted that the basic asymmetry of the right skewed N-period terminal wealth distribution exhibited in (10) is not reflected in the asymptotic compound return distribution. This fact is one of the fundamental differences between the single-period and multi-period portfolio selection problems. In the single-period case, the return distribution and terminal wealth distribution have the same essential characteristics. In the multi-period case, the return distribution is asymptotically symmetric, while the wealth distribution is right skewed. We cannot ignore the possibility that, for a particular investor, the right skew of the N-period terminal wealth distribution may have high utility.

We have shown that the mean of the compound return distribution is asymptotically directly related to the median of the N-period terminal wealth distribution. In highly skewed distributions, the median is often the descriptive parameter of choice. The mean of the N-period terminal wealth distribution which is dependent on large but unlikely events, may, for many purposes, be virtually irrelevant as a description of the probability distribution. Therefore, the mean (and median) of the geometric mean can be an excellent summary statistic for a multi-period investor, relating N-period terminal wealth to the context of single-period returns, in cases where the median of the N-period terminal wealth distribution is an appropriate description of the distribution. An investor with a safety first criterion (Roy, 1952) based on compound return may find the mean and variance of geometric mean return a particularly appropriate basis for multi-period portfolio selection (c.f. Section 5).

3. The mean-variance compound return portfolio selection model: Assumptions and derivations.

An approximation of the geometric mean return is used to compute the mean and variance of the geometric mean return distribution, when the number of periods is finite, under the assumption that single-period returns are approximately normally distributed. A number of approximations exist which express the geometric mean as a function of the moments of the returns (Young and Trent, 1960). For purposes of mathematical tractability, simplicity of derived results, and consistency with the mean-variance description of single-period returns, the one we shall use is:

$$G_N(R) = \bar{r} - \frac{s^2(R)}{2} \tag{11}$$

where $\bar{r}$ is the average and $s^2(R)$ is the (biased) sample variance of the given returns.

Young and Trent have shown that (11) can be an accurate estimate of the geometric mean of portfolio returns. It is evident, however, that for (11) to be useful, the distribution of returns must be well described by the mean and variance.

A good example of (11) as a poor estimate of the geometric mean is given by Hakansson(1971, pp. 526-529). He shows that a portfolio with maximum long term expected geometric return may not be mean-variance efficient. This paradox can be explained by observing the high skew in Hakansson's portfolio. The paradox disappears if we apply mean-variance portfolio selection techniques to the case where the mean and variance adequately describes the distribution of portfolio returns.

Apart from our assumptions concerning preferences, and the highly standardized premises in single-period portfolio theory such as no transaction costs and taxes, we also assume portfolio rebalancing and a unit value adjustment of wealth for computing compound return at the start of each period of the N-period investment horizon. Single-period returns are assumed generated consistent with the return-risk relationships (1) and (3) in a non-equilibrium capital market. The necessary return distribution assumptions required for the mathematical derivations which follow are intertemporal independence and a stationary investment opportunity set. To compute the variance of the geometric mean in terms of the mean and variance, we will assume that the third central moment of single period returns is equal to zero and the fourth central moment is equal to three times the square of the variance.

The assumption of a stationary investment opportunity set[3] and constant investment policy over the investment horizon is primarily a mathematical convenience. The mathematical results remain essentially unaltered under a variety of non-stationary intertemporal assumptions.

[3]Appendix A provides a derivation of this section's results when R_o is a random variable. This will be appropriate if R_o varies intertemporally or is a single-period random variable according to the Black (1972) model for risky assets.

The expected value for the geometric mean return (11) can be computed using standard statistical techniques (e.g. Hogg and Craig, 1973). It then follows from our assumptions that the expected N-period compound return is equal to

$$\begin{aligned} E(G_N(R)) &= E(\bar{r}) - E(s^2)/2 \\ &= \mu - (1-\tfrac{1}{N})\,\frac{\sigma^2}{2} . \end{aligned} \tag{12}$$

Using equations (1) and (3), the expected value of $G_N(R)$ can be written in terms of the single-period investment policy and capital market parameters[4]:

$$E(G_N(R))=R_o+\beta(E(M)-R_o)-(1-\tfrac{1}{N})\beta^2\sigma_M^2/2\rho^2. \tag{13}$$

Proceeding in the same way, we can derive the variance of $G_N(R)$ as follows (see Fisz, 1963, p. 369, 9.2 and p. 371, 9.17):

$$\begin{aligned} V(G_N(R)) &= V(\bar{r}) + V(s^2)/4 \\ &= \frac{\sigma^2}{N}(1 + (1-\tfrac{1}{N})\,\frac{\sigma^2}{2}) \end{aligned} \tag{14}$$

which can be written as

$$V(G_N(R)) = \frac{\beta^2\,\sigma_M^2}{\rho^2 N}\quad(1+(1-\tfrac{1}{N})\,\frac{\beta^2\sigma_M^2}{2\,\rho^2}). \tag{15}$$

Equation (13), the N-period geometric mean security market "line", specifies N-period expected compound return as a function of portfolio risk (β).

[4]For our purposes, the market environment is characterized by the capital market parameters: expected market return, standard deviation of market returns, and the risk free rate.

For N>1, the formula for expected compound return (13) is a quadratic function of beta with a peak[5] at

$$\beta_{C,N} = \frac{(E(M)-R_o)\ \rho^2}{(1-\frac{1}{N})\ \sigma^2_M} . \tag{16}$$

$\beta_{C,N}$ is the market risk policy which maximizes expected compound return over N investment periods.

Upon taking the limit of (13) as $N\to\infty$, we have the expected compound return for a constant risk and diversification strategy over an infinite number of investment periods:

$$\lim_{N\to\infty} E(G_N(R)) = E^*(G(R)) = R_o + \beta(E(M)-R_o) - \frac{\beta^2\sigma^2_M}{2\rho^2} \tag{17}$$

[5]The accuracy of the mathematical results of this section is dependent on the approximative power of (11). For values of the capital market parameters which typically characterize historical data (e.g., Table 8.1 in Sharpe, 1970, p.148), the approximation is sufficiently accurate for most purposes and provides a clear understanding of the basic characteristics of the N-period mean-variance geometric mean return problem. However, an important exception is that (16) fails to show that a maximum beta for expected geometric mean return may not exist. In Appendix B the basic results of this section are rederived for two other more accurate approximations of the geometric mean. As can be seen from the appendix solutions, the non-existence of a maximum or critical beta occurs for values of the capital market parameters which indicate an optimistic capital market environment. The resulting increase in accuracy of the appendix approximations required further assumptions and resulted in an increase in the complexity of the solutions. In practical terms, the differences between the three solutions are usually small for typical values of the capital market parameters. If the critical beta in (16) is large, the possible non-existence of a critical beta should be checked using the formulas in the appendix. In the following, we will assume the existence of a critical beta, unless otherwise noted.

Equation (13) shows that expected compound return decreases with time and increases with increasing diversification ($N>1$). The formula in (17) provides a lower bound for expected compound return over time for given values of the investment policy parameters β and ρ.

The critical value of beta defined in (16) denotes not only an optimal expected compound return risk level, but also a maximum sensible level of market risk in the given market environment. To illustrate this we observe that the variance in (15) is an increasing function of market risk. Then for β greater than (16) there is a $\beta' < \beta_{C,N}$ such that the compound return at β' will be greater than or equal to the return at β with less variance.

The values of the investment policy parameters which jointly maximize expected compound return in (13) and (17) are:

$$\begin{aligned} \rho^* &= 1 \\ \beta^*_{C,N} &= (E(M)-R_o)/(1-\tfrac{1}{N})\sigma_M^2 \\ \beta^*_C &= (E(M)-R_o)/\sigma_M^2 \; . \end{aligned} \tag{18}$$

Evaluating (17) at β^*_C and ρ^* we have:

$$\max E^* \; (G(R)) = R_o + (E(M)-R_o)^2/2\sigma_M^2. \tag{19}$$

This formula represents the maximum (with probability one) obtainable compound return, or characteristic market return, from investing in the given capital market, on a long term basis.

From (12) the long run portfolio growth rate in terms of the mean and variance of single period returns is:

$$\lim_{N\to\infty} E(G_N(R)) = \mu-\sigma^2/2. \tag{20}$$

Although (20) is not always a good estimate of (6), for consistency and mathematical convenience we will continue to use (17), and by implication (20), as an estimate of the long term growth rate, rather than an approximation based directly on (6)[6].

By definition, portfolios on the one-period capital market line are perfectly correlated with the market portfolio. An examination of (13) and (15) will show that for a given value of β, $E(G_N(R))$ increases and $V(G_N(R))$ decreases as $\rho\to 1$. Hence, the N-period geometric mean capital market line will consist of portfolios on the one-period capital market line. The set of N-period efficient portfolios will lie on a segment of an approximately quadratic curve in $(E(G_N(R)), \sigma^2(G_N(R)))$space. The effect of the N-period analysis is to confine the set of N-period mean-variance of compound return efficient portfolios to a subset of the one-period mean-variance efficient set. In terms of the one-period capital market line, the standard deviation of N-period efficient portfolios ranges from zero to

$$\sigma_{Max}=\beta^*_{C,N}\sigma_M. \tag{21}$$

[6]The approximation of the geometric mean in Appendix B.2 has the correct long term growth rate (6).

Equation (13) predicts that the portfolio risk-average geometric mean return relationship should be approximately quadratic over an N-period investment horizon and that, over long time periods and beyond a certain risk level, average portfolio compound return and median wealth decreases. Most studies which have reported a linear relationship between beta and portfolio return (e.g. Black, Jensen, and Scholes, 1972) examined data over short time periods, usually monthly. The work of Pratt (1971) and Blume and Friend (1974) provide some empirical support for our results.

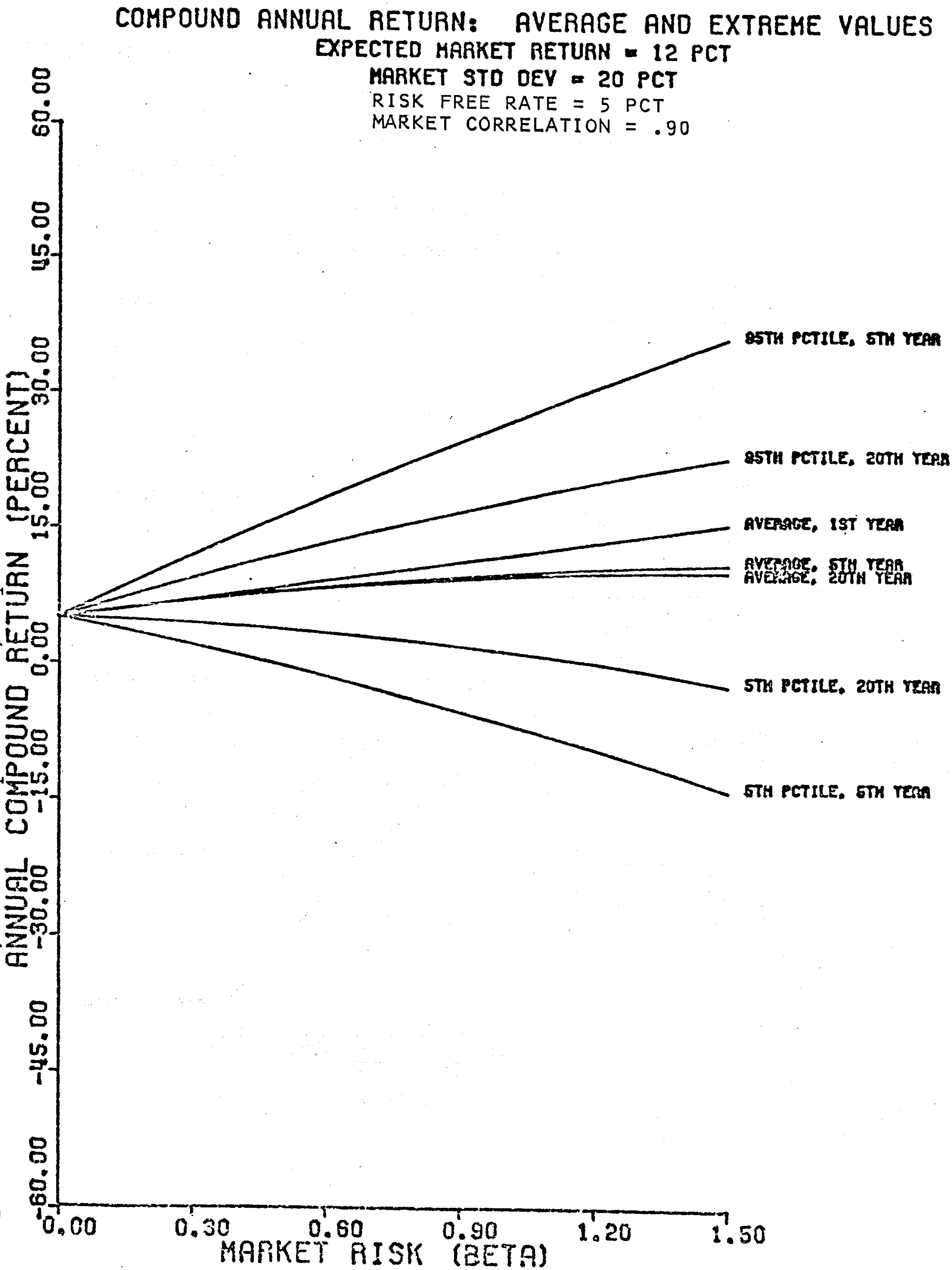
4. Compound return and portfolio management.

In Fig. 1 which follows, a display of the market risk-expected compound return relationship given by equations (13) and (15) is shown for a portfolio with a correlation of 0.9 with the market. An expected yearly (one-period) market return of 12%, a standard deviation of 20% and a risk-free rate of 5% were assumed. The values of the capital market parameters used in the following figures are only illustrative and are not intended to represent any prediction of future market performance.

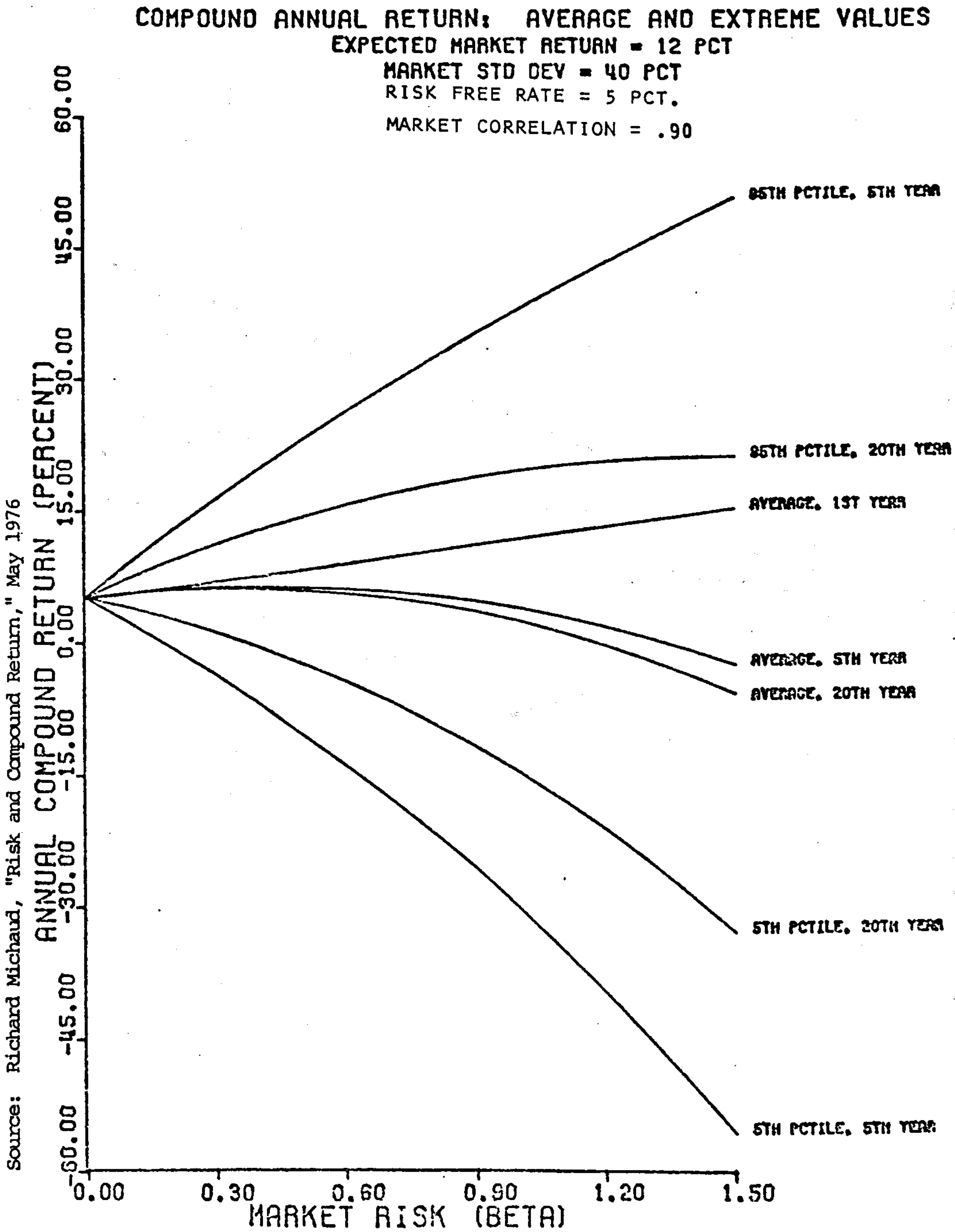
For this case, average or expected compound return does not increase substantially for $\beta > 1$. On a more general level, the curves labeled "average" for the indicated years demonstrate a general geometric mean return principle: high-risk return melts away over time. The substantial gains in return which will, on average, accrue to high-risk portfolios on a one-period basis will tend to diminish over a sufficiently long investment horizon.

The curves labeled 5th and 95th percentile are derived on the basis of the assumption that the geometric mean return is normally distributed. They indicate that less than 5 percent of returns will be below a 5th percentile curve in the given year, while less than 5 percent of returns will be greater than the 95th percentile.

Fig. 2 illustrates the effect on compound return of investing in a considerably riskier capital market environment (higher market standard deviation). The downward sweep of the average compound



Source: Richard Michaud, "Risk and Compound Return," May 1976.



COMPOUND ANNUAL RETURN: AVERAGE AND EXTREME VALUES
EXPECTED MARKET RETURN = 12 PCT
MARKET STD DEV = 40 PCT
RISK FREE RATE = 5 PCT.
MARKET CORRELATION = .90
95TH PCTILE, 5TH YEAR
95TH PCTILE, 20TH YEAR
AVERAGE, 1ST YEAR
AVERAGE, 5TH YEAR
AVERAGE, 20TH YEAR
5TH PCTILE, 20TH YEAR
5TH PCTILE, 5TH YEAR
ANNUAL COMPOUND RETURN (PERCENT)
60.00
45.00
30.00
15.00
0.00
-15.00
-30.00
-45.00
-60.00
0.00
0.30
0.60
0.90
1.20
1.50
MARKET RISK (BETA)
Source: Richard Michaud, "Risk and Compound Return," May 1976

Fig. 2

return curves demonstrates the fact that the quality of the capital market environment has an important effect on the compound return distribution. We also note, from an examination of (13) and (16), that reducing the risk premium $(E(M) - R_o)$ has an effect similar to that of increasing the market variance.

One way to interpret and understand these results is to recognize that negative return has a greater impact on terminal wealth than positive return. Therefore, if both large positive and large negative returns are possible, as they are with high values of beta or in risky markets, the average compound rate of return may begin to decrease relative to a less risky portfolio.

Constant risk and diversification strategies can be understood in terms of their expected compound return as a function of time. We have observed from (13) that expected compound return decreases with time and that (17) is the ultimate (with probability one) compound return of a constant risk and diversification policy. The peak of the curve (17) as a function of beta occurs at

$$\beta^*_p = (E(M)-R_o)/(\sigma_M^2/\rho^2). \tag{22}$$

β^*_p is the value of beta for an optimal long term constant risk and diversification policy.

Although expected return for a portfolio with $\beta > \beta^*_p$ will have higher expected initial value than for the β^*_p portfolio, the effect of compounding, which decreases expected compound return, will be greater for the β portfolio. Given a sufficiently long period of time -- often as little as three periods -- the expected compound return from the high beta portfolio will be less than for the β^*_p portfolio. The value of N for which this crossover effect occurs is a decreasing function of beta.

In Fig. 3, the expected compound return-time relationship is displayed for the indicated values of beta, including beta* the optimal long term constant risk policy from (22). The market and correlation parameters correspond to those of Fig. 1.

In the relatively low risk (or high risk premium) market of Fig. 3, $\beta^*_p=1.42$. Thus increasing market risk not only raises initial expected return, but also increases expected long term compound return as well, up to the critical value of 1.42.

In Fig. 4, the expected compound return-time relationship is presented for the riskier capital market case corresponding to the market and correlation parameters of Fig. 2. This figure illustrates the significant deterioration of expected compound return that is likely in a high risk market. Rising market risk lowers the ultimate expected portfolio compound return, even as it increases initial expected return, for any value of beta

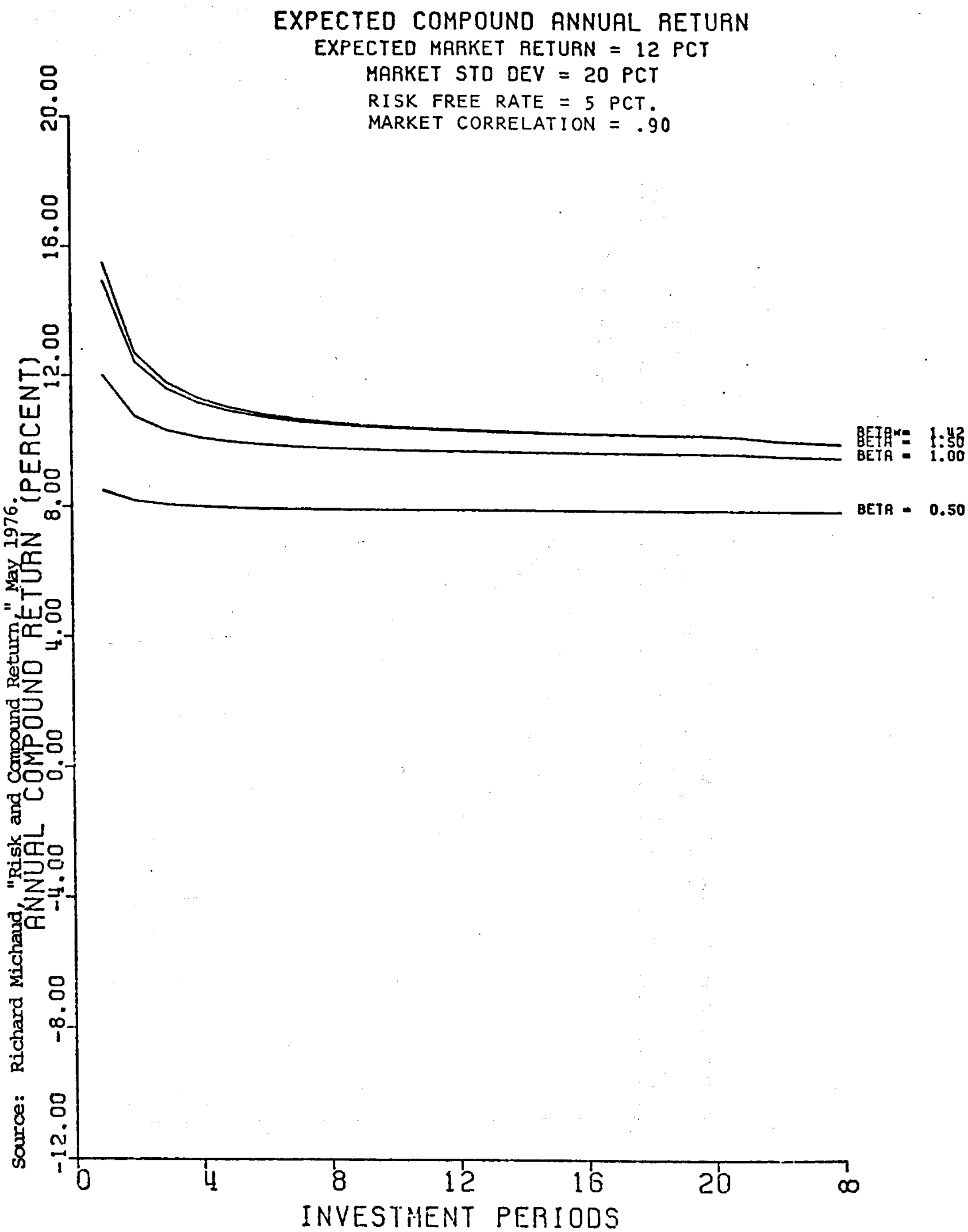

EXPECTED COMPOUND ANNUAL RETURN
EXPECTED MARKET RETURN = 12 PCT
MARKET STD DEV = 20 PCT
RISK FREE RATE = 5 PCT.
MARKET CORRELATION = .90
ANNUAL COMPOUND RETURN (PERCENT)
20.00
16.00
12.00
8.00
4.00
0.00
-4.00
-8.00
-12.00
0
4
8
12
16
20
∞
INVESTMENT PERIODS
BETA = 1.42
BETA = 1.50
BETA = 1.00
BETA = 0.50
Source: Richard Michaud, "Risk and Compound Return," May 1976.

Fig. 3

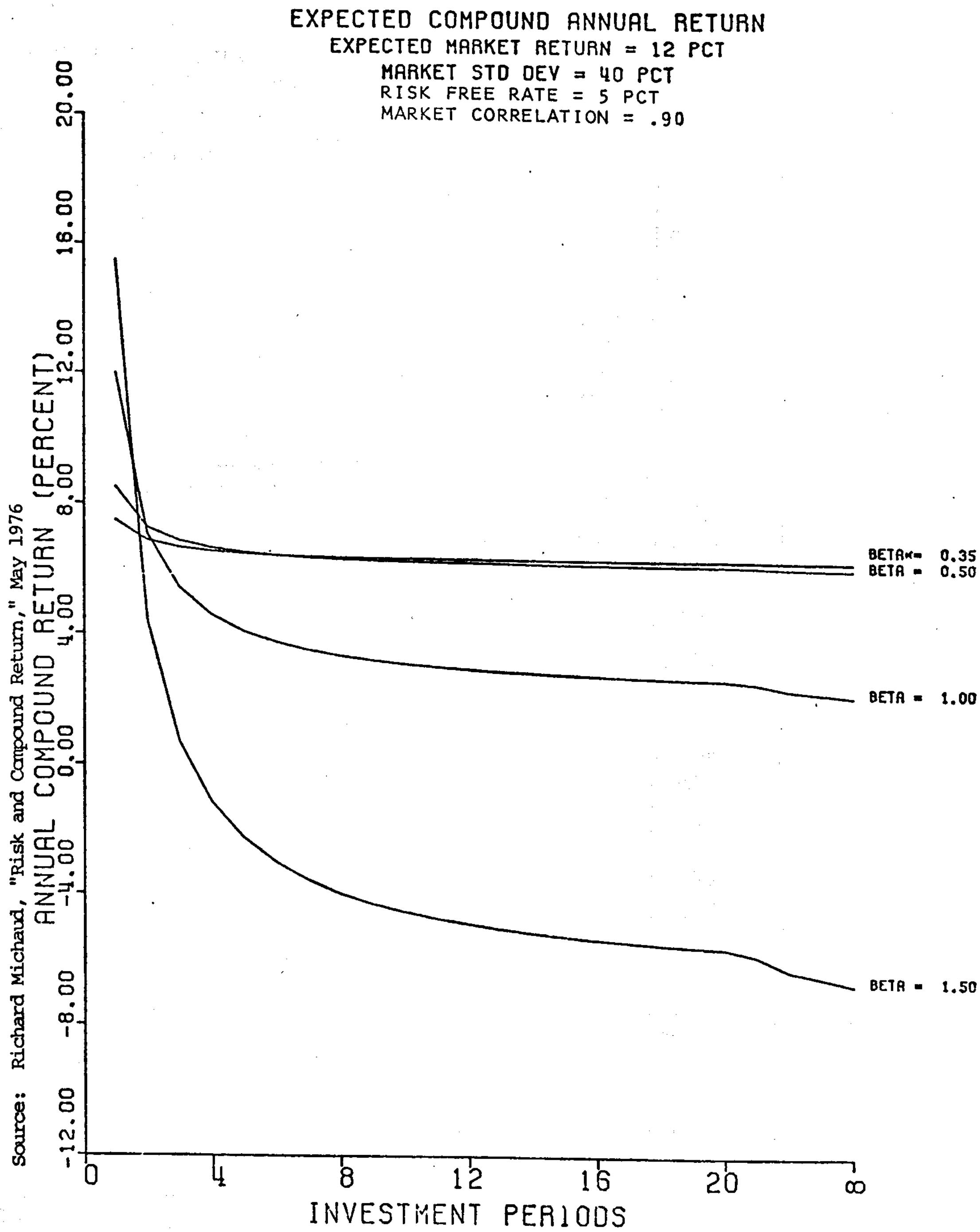

EXPECTED COMPOUND ANNUAL RETURN
EXPECTED MARKET RETURN = 12 PCT
MARKET STD DEV = 40 PCT
RISK FREE RATE = 5 PCT
MARKET CORRELATION = .90
ANNUAL COMPOUND RETURN (PERCENT)
20.00
16.00
12.00
8.00
4.00
0.00
-4.00
-8.00
-12.00
0
4
8
12
16
20
∞
INVESTMENT PERIODS
BETA = 0.35
BETA = 0.50
BETA = 1.00
BETA = 1.50
Source: Richard Michaud, "Risk and Compound Return," May 1976

Fig. 4

greater than 0.35[7]. Although the preceding analysis has been discussed within the context of constant risk strategies, various non-stationary parameter assumptions, assuming intertemporal independence of market returns, leave the essential results unchanged.

Using the characteristic market return formula (19), we can compare constant risk policies across markets. Suppose M_A and M_B denote total returns from two capital markets such that B is a higher risk-higher return market; i.e.,

$$E(M_B) > E(M_A)$$

$$\sigma_{M_B} > \sigma_{M_A}$$

but the characteristic return (19) from A is greater than for B. Although initial expected return, for optimal constant risk policies, will be greater in the first period for B than for A, long-term expected compound return will be smaller.

The value of β^*_C in (18) is such that, on a sufficiently long-term basis, any portfolio with $\beta>\beta^*_C$ will have less return and more risk. Since β^*_C is dependent on capital market parameters alone, it can be used to characterize the relative riskiness of the given market.

[7] In order to determine the accuracy of our analytic results, Monte Carlo simulations were performed over a twenty-period investment horizon for various values of the capital market parameters. Each period's return was generated using the IBM-SSP Gauss subroutine. Returns were left truncated and accumulated at minus one. The results showed that (16) was a good estimate of the critical beta and that (13) was, in general, a good approximation of expected compound return. When there was a significant difference, we found the analytic solution tended to overstate expected compound return. In general (15) tended to underestimate the variance of compound return.

5. Compound return probability.

Estimates of compound return probabilities often have been considered of primary importance in long range investment planning and have been cited as a major reason for conducting a portfolio simulation study (Williamson, 1970, p.83). Using advanced statistical techniques (Rao, 1965, p. 321(ii) and Fisz, 1963, 9.20) we can show that the distribution of (11) approaches a normal distribution as $N \to \infty$. Using the limiting distribution of (11) as justification, we consider an approximate relationship between risk policy and compound return probability for any investment horizon and level of compound return or N-period terminal wealth.

We will assume that

$$P(G_N(R) > K) = 1 - \phi(z) \tag{23}$$

where $\phi(z)$ is the cumulative normal distribution of the standard normal random variable z, and where

$$z = \frac{K - R_o - \beta(E(M) - R_o) + (1-\frac{1}{N}) \dfrac{\beta^2 \sigma_M^2}{2\rho^2}}{\dfrac{\beta \sigma_M}{\rho\sqrt{N}} \sqrt{1 + (1-\frac{1}{N}) \dfrac{\beta^2 \sigma_M^2}{2\rho^2}}} \,. \tag{24}$$

We note that as $N \to \infty$, the probability (23) is either one or zero depending on whether the required rate of return K is less than or greater than $E^*(G(R))$ in (19).

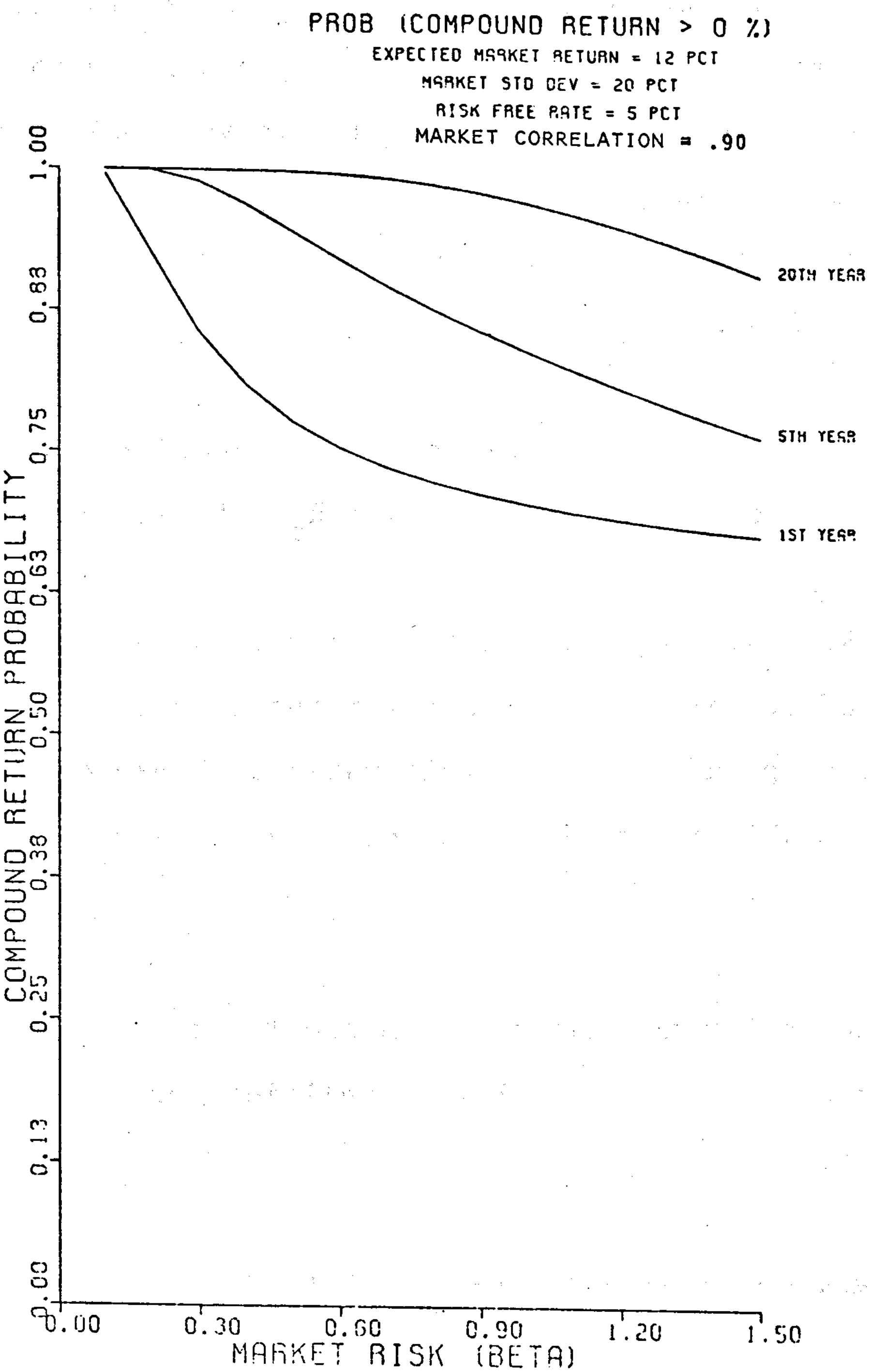
The relationship of risk and compound return probability (23) for achieving at least a 0% compound return for investment horizons of one, five, and twenty investment periods (years) is illustrated in Fig. 5. The capital market parameters are those given in Figs. 1 and 3. Fig. 5 shows that on a one-year basis, the effect of increasing market risk is to reduce the probability that a 0% return level will be achieved. The increase in variability in the portfolio dominates increasing return. The compounding effect of time reduces the variability of return which, in this case, increases the probability.

Fig. 6 illustrates the effect of a reduced risk premium on compound return probability, where $K > R_o$. First year compound return probability is considerably less than the corresponding probability in Fig. 5. The compound effect of time, which decreases variance, is not sufficient to compensate for the inherent high risk and lowered expected compound return in this market. Therefore, the compound return probability in the twentieth year is lower than in the first year.

In the following subsections A and B, we state and solve two problems concerning optimal risk policies and compound return probabilities.

A. Maximum compound return probability.

By minimizing z in (23), as a function of β, we can find the risk policy, for any investment horizon, which maximizes the probability that a given compound rate of return or wealth level will be achieved.



Source: Richard Michaud, "Risk and Compound Return," May 1976

Fig. 5

Source: Richard Michaud, "Risk and Compound Return," May 1976

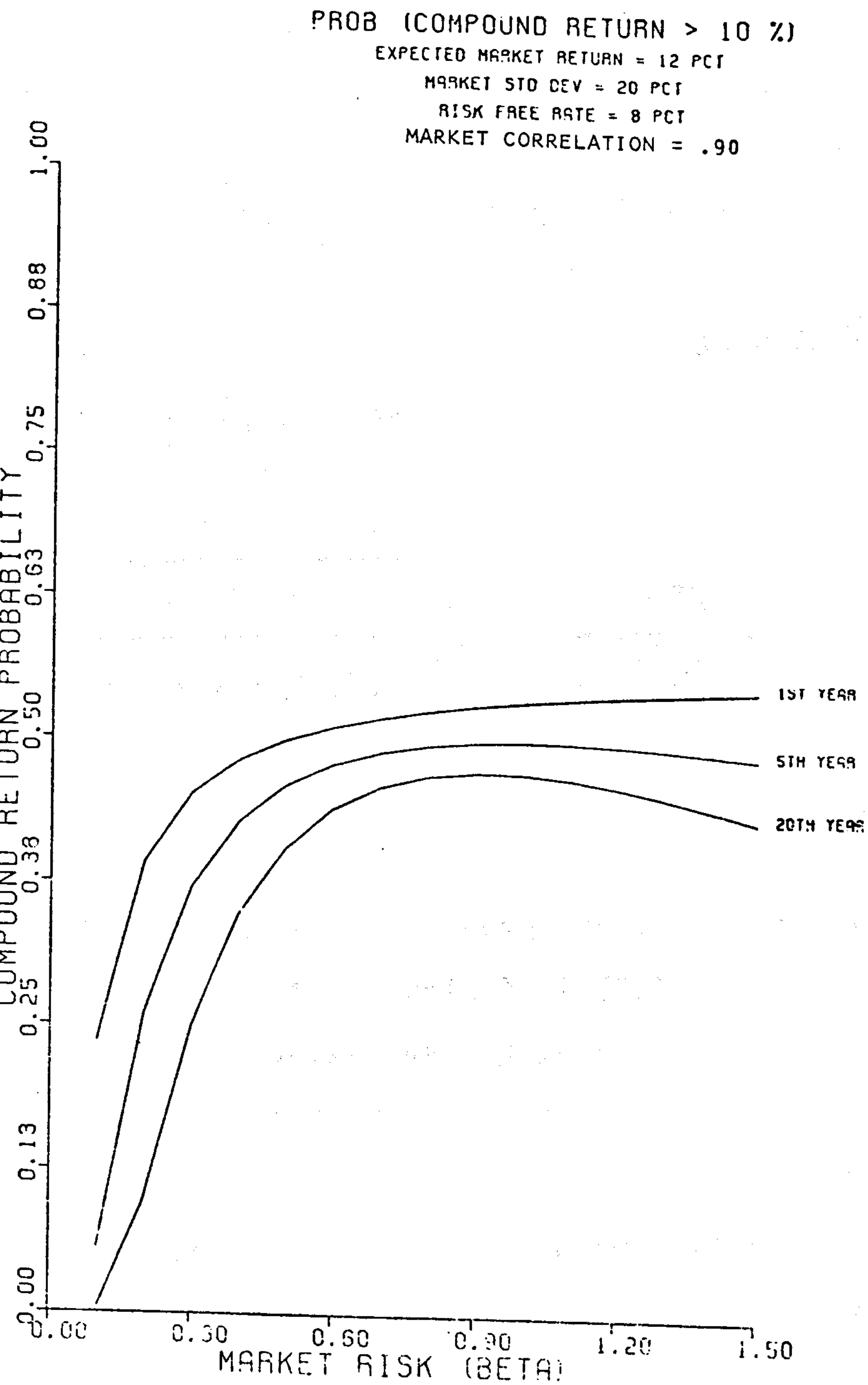


Fig. 6

The resulting condition for a minimum is a cubic equation in β:

$$R_o - K + (1-\tfrac{1}{N})\ \frac{\sigma_M^2}{2\rho^2}\ (1-2(K-R_o))\beta^2$$

$$+\ (E(M)\ -\ R_o)(1-\tfrac{1}{N})\ \frac{\sigma_M^2}{2\rho^2}\ \beta^3 = 0. \tag{25}$$

The following iteration formula for finding the β which maximizes compound return probability ($\beta^*_{K,N}$), has been found to be useful:

$$\beta = \frac{\sqrt{2}\ \rho}{\sigma_M\sqrt{1-\frac{1}{N}}}\ \sqrt{\frac{K\ -\ R_o}{(E(M)\ -\ R_o)\beta+1+2(R_o-K)}}\ . \tag{26}$$

For parameters of interest, the denominator in the radical (26) should be positive. Therefore, a solution exists only if $K > R_o$. If $K < R_o$, the compound probability is at its maximum 1.0, when $\beta^*_{K,N} = 0$. From (25) it follows that $\beta^*_{K,N}$ is a decreasing function of the return level K. By taking the limit as $N\to\infty$, we can find, using (25) or (26), β^*_K, which is the long run dominant risk policy for the compound return probability (23) for given compound rate of return K. Using (26), the peaks of the five and twenty year probability curves in Fig. 6 occur at 1.01 and 0.92 respectively.

B. Maximum K for a specified compound probability.

By our assumption of normality of $G_N(R)$, if $K=E(G_N(R))$, the probability (23) is 0.50. Therefore, the probability of achieving or surpassing the maximum expected growth rate over the investment horizon following a critical beta policy is 0.50. An investor may be willing to forego a high growth rate in order to achieve a growth rate with a specified probability. We will now solve the problem of finding the risk policy which maximizes the growth rate K in (23) subject to a specified probability level P_o. This problem amounts to maximizing compound return with a probability side condition.

In mathematical notation, we seek to maximize K as a function of β, where K satisfies the relation

$$P(G_N(R) > K\ (\beta)) = P_o \tag{27}$$

and P_o is a given specified probability level.

Note that, if $P_o = .90$, then $\phi(z_o) = .10$, where z_o is the value of the standard normal variable. Therefore z_o, in the above example, will be negative. For convenience, we will use $-z_o$, so that the correspondence between P_o from tables of values of standard normal cumulative probabilities and the probability P_o is direct.

Using (24), with $-z_o$ replacing z, we can write K as a function of β as:

$$K(\beta) = R_o + \beta(E(M)-R_o) - (1-\frac{1}{N}) \frac{\beta^2\sigma_M^2}{2\ \rho^2} - \frac{z_o\ \sigma_M\beta}{\rho\ \sqrt{N}} \sqrt{1+(1-\frac{1}{N}) \frac{\beta^2\sigma_M^2}{2\rho^2}} \tag{28}$$

Taking the derivative of K(β) with respect to β and setting the equation to zero, we have a fourth degree equation in β which must be satisfied for a risk policy which maximizes compound return or wealth, subject to the probability relation (27). Instead of writing the fourth degree equation, we give an iteration formula for the solution which more clearly describes the functional relationships of the optimal risk policy with the values of the various parameters:

$$\beta = \frac{(E(M)-R_o)\rho^2}{\sigma_M^2\ \ (1 - \frac{1}{N})} - \frac{z_o\ \ \rho(1+(1-\frac{1}{N})\frac{\beta^2\sigma_M^2}{\rho^2})}{\sigma_M\ \sqrt{N}\ \ (1-\frac{1}{N})\sqrt{1+(1-\frac{1}{N})\beta^2\sigma_M^2/2\rho^2}} \tag{29}$$

Let $\beta^*_{P_o,N}$ denote the risk policy which maximizes the growth rate that can be achieved in a given investment horizon, for a specified probability level P_o. Using (28) we note that $\beta^*_{P_o,N} \to \beta^*_C$ when $N\to\infty$. Also, using (29), $\beta^*_{P_o,N} = \beta_{C,N}$ when $P_o = 0.50$ and

$$\beta^*_{P_o,N} \doteq \frac{(E(M)-R_o)\rho^2}{\sigma^2_M(1-\frac{1}{N})} - \frac{z_o\rho}{\sigma_M\sqrt{N}} . \tag{30}$$

Since a growth rate equal to the risk free-rate can be assured with probability equal to one, then $\beta^*_{P_o,N} \geq 0$. If P_o is sufficiently high, $K(\beta)$ in (28) may have a maximum and (29) may have a solution at a negative β.

From an analysis of (30), for parameters of interest, the condition

$$z_o < \frac{(E(M)-R_o)\rho\sqrt{N}}{\sigma_M} \tag{31}$$

is necessary and nearly sufficient for $\beta^*_{P_o,N}>0$.

Inequality (31) is of some theoretical interest. Not every probability level P_o is attainable with a positive risk policy. For conservative investors who require a high probability of assurance, P_o, the only appropriate risk policy may be zero risk. On the other hand, the presence of the $\sqrt{N}$ factor in (31) indicates that long term investment horizons can assure high probability levels concerning compound rates of return and levels of N-period terminal wealth.

6. Summary and conclusions.

From considerations of the geometric mean return distribution we have shown that: 1) the distribution is asymptotically normal, and therefore that the mean and variance are asymptotically the appropriate descriptive parameters of the distribution; 2) expected compound return is a decreasing function of the number of periods; 3) the quantiles of the geometric mean return distribution are directly related to the N-period right skewed terminal wealth distribution. In particular, this implies that the mean of compound return is (asymptotically) directly related to the median of N-period terminal wealth. For multi-period risk averse maximizers of the expected utility of N-period terminal wealth,such that the median is an appropriate descriptive parameter of the N-period terminal wealth distribution,or for investors with (multi-period) safety first objectives, the mean and variance of compound return can be a useful criterion for multi-period portfolio selection.

A mean-variance approximation of the geometric mean, in conjunction with the Fama (1970) analysis and the single-period return-risk relationships (1) and (3), was used in order to derive a model of multi-period portfolio performance. It also allowed us to determine efficient single-period investment policy portfolio decisions for a risk-averse multi-period investor who chooses among alternative portfolios on the basis of the mean and variance of compound return. The major results of our analysis are: 1) the N-period compound return security market "line" (13) is approximately quadratic as a function of market risk; 2) a critical beta (16) generally exists which is a maximum sensible risk policy; 3) the N-period set of efficient mean-variance compound return portfolios is a subset of the single-period mean-variance efficient set; 4) a formula for the maximum long term growth rate obtainable (with probability one) to any investor in a given capital market (19) can be used to characterize capital markets or to compare one market with another.

By assuming that compound return probability is normally distributed when N is finite, risk policies were found which maximize: a) the probability that a given growth rate will be achieved over the N-period investment horizon; b) the growth rate that can be achieved in an N-period investment horizon for a specified probability level.

The objective of the analysis has been to clarify the long term risk-return relationship for rational portfolio decision making. This has led to restrictions on the (one-period) efficient investment opportunity set and to new tools for analyzing the consequences of a long term investment policy. An N-period risk-averse expected utility maximizer of N-period terminal wealth is thus supplied with a clearer basis on which to determine an optimal investment policy.

For portfolio management, perhaps the most important aspect of this analysis is the awareness of the limitations of high beta securities and portfolios, and the possible suboptimality of fully invested market-like portfolios, such as mutual funds, for investors with long range investment objectives. Concomitantly, any long term assessment of investment policy should recognize the critical role of market parameter assumptions on the decision making process.

Appendix A: Compound return mean-variance analysis when R_o is a random variable.

We assume that the single-period return generation process for a security or portfolio is consistent with

$$E(R) = E(R_o)(1-\beta) + \beta\ E(M)$$

$$\sigma_R^2 = \frac{\sigma_{R_o}^2\ (1-\beta)^2 + \beta^2\sigma_M^2 + 2\beta(1-\beta)COV(R_o,M)}{\rho_1^2} \tag{1A}$$

where ρ_1 is a multiple correlation. It follows that

$$E(G_N(R)) = E(R_o) + \beta(E(M)-E(R_o))$$

$$\frac{(1-\frac{1}{N})(\sigma_{R_o}^2\ (1-\beta)^2+\beta^2\sigma^2{}_M + 2\beta\ (1-\beta)COV(R_o,M))}{2\rho_1^2} \tag{2A}$$

$$V(G_N(R)) = \frac{(\sigma^2{}_{R_o}\ (1-\beta)^2+\beta^2\sigma^2{}_M + 2\beta(1-\beta)COV(R_o,M))}{\rho_1^2\ \ N}$$

$$\cdot\{1+\frac{(1-\frac{1}{N})}{2\rho_1^2}\ \ (\sigma_{R_o}^2\ (1-\beta)^2+\beta^2\sigma^2{}_M + 2\beta(1-\beta)COV(R_o,M))\} \tag{3A}$$

$$\beta_{C,N} = \frac{(E(M)-E(R_o)\rho_1^2}{(1-\frac{1}{N})(\sigma^2_{R_o} + \sigma^2_M - 2COV(R_o,M))} + \frac{\sigma^2_{R_o} - COV\ (R_o,M)}{\sigma^2_{R_o} + \sigma^2_M - 2COV(R_o,M)} \qquad (4A)$$

If $COV(R_o,M) = 0$, then in general, the net effect of the introduction of R_o as a random variable with respect to the assumed return generation process is to decrease expected compound return (2A), increase the variance of compound return (3A) and decrease the critical value of beta (4A), with respect to the corresponding derivations in Section 3.

Appendix B: Alternative approximations of the geometric mean.

1) Using Taylor's formula for N variables, expanding (4) about the point $(\mu,\mu \ldots ,\mu)$ and ignoring terms of order three and higher or those with zero expectation we derive the approximation:

$$G_N(R) \doteq \bar{r} - \frac{s_o^2 \, (1-\frac{1}{N})}{2(1+\mu)} \tag{1B}$$

where s_o^2 is defined as

$$s_o^2 = \frac{\Sigma(r_i-\mu)^2}{N}.$$

Then

$$E(G_N(R)) = R_o + \beta(E(M)-R_o) - \frac{(1-\frac{1}{N}) \; \beta^2 \; \sigma_M^2}{2\rho^2(1+R_o+\beta(E(M)-R_o))} \tag{2B}$$

$$V(G_N(R)) = \frac{\beta^2\sigma_M^2}{\rho^2 N} \left(1+ \frac{(1-\frac{1}{N}) \quad \beta^2\sigma_M^2}{2\rho^2(1+R_o+\beta(E(M)-R_o))^2}\right) \tag{3B}$$

$$\beta_{C,N} = \frac{(1+R_o)}{(E(M)-R_o)} \left(\frac{1}{\sqrt{1- \dfrac{2\rho^2(E(M)-R_o)^2}{\sigma_M^2(1-\frac{1}{N})}}} -1 \right) \tag{4B}$$

From an analysis of (4B) it can be shown that $\beta_{C,N}$ has, in general, the functional characteristics of (16). A major difference is that a critical beta may fail to exist in (4B) if the risk premium is sufficiently large or market variance sufficiently small. Generally, expected compound return is larger, variance smaller and the critical beta greater than the corresponding derivations in Section 3.

Equations (1B) to (4B) will be more accurate (statistically efficient) than corresponding estimates in Section 3, since more prior knowledge was assumed in their derivation. They may prove useful in portfolio simulation studies since μ is a given.

2. An alternative approximation of $E(G_N(R))$ will be derived which has the correct limit (6) as $N \to \infty$.

For r suitably constrained, we write the binominal series expansion

$$(1+r)^{\frac{1}{N}} = 1+\frac{r}{N} - \frac{(1-\frac{1}{N})r^2}{2N} + \frac{(1-\frac{1}{N})}{2}\frac{(2-\frac{1}{N})}{3N} r^3 + \ldots$$

$$\doteq 1+\frac{r}{N} + \frac{(1-\frac{1}{N})}{N}\left(-\frac{r^2}{2} + \frac{r^3}{3} - \ldots \right)$$

$$= 1+\frac{r}{N^2} + (1-\frac{1}{N}) \frac{\ln(1+r)}{N} . \qquad (5B)$$

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From (5B) and (4) the geometric mean can be approximated by

$$G_N(R) \doteq \prod_{i=1}^{N} \left(1 + \frac{\ln(1+r_i)}{N} + \frac{r_i - \ln(1+r_i)}{N^2}\right) - 1. \tag{6B}$$

Hence, assuming intertemporal independence and identically distributed returns,

$$E(G_N(R)) = \left(1 + \frac{E(\ln(1+r))}{N} + \frac{E(r) - E(\ln(1+r))}{N^2}\right)^N - 1 \tag{7B}$$

which can be shown to have the limiting value (6) as $N \to \infty$.

Using the fact that

$$E(\ln(1+r)) \doteq \ln(1+\mu) - \frac{\sigma^2}{2(1+\mu)^2}$$

for values of μ and σ which typically characterize single-period returns, we can approximate (7B) in terms of the mean and variance of returns in each period:

$$E(G_N(R)) = \left(1 + \frac{\ln(1+\mu) - \frac{\sigma^2}{2(1+\mu)^2}}{N} + \frac{\mu - \ln(1+\mu) + \frac{\sigma^2}{2(1+\mu)^2}}{N^2}\right)^N - 1. \tag{8B}$$

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