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RISK POLICY AND LONG-TERM INVESTMENT

*Richard O. Michaud**

I. Introduction

Empirical tests of the Sharpe [36]-Lintner [23]-Black [3] Capital Asset Pricing Model (CAPM) have generally concluded that there is a positive, approximately linear, trade-off between average return and systematic risk (beta) for portfolio returns of common stocks. Most of the empirical studies, however, have reported data for short, usually monthly, time intervals.[1] Exceptions to this rule include Blume and Friend [8] and Sharpe [38, pp. 289-292]. Their data provide evidence that long-term wealth ratios are concave, possibly nonmonotonic, functions of beta. These data are surprising since, if returns are intertemporally independent and the linear return model of CAPM is correct, expected multiperiod terminal wealth is a convex, monotone increasing function of beta. The results of this paper provide a theoretical framework for interpreting the long-term empirical data which does not violate the notion of a monotone increasing expected terminal wealth-beta relationship.

If empirical evidence of the *ex post* behavior of common stock portfolios is consistent with market efficiency and a linear positive relationship of expected return with systematic risk (Jensen [20]) in each period,[2] then the CAPM parameters describing the level of systematic risk and diversification provide empirically and theoretically relevant descriptions of normative investment policy. In perfect and efficient capital markets (Fama [10]), the choice of the level of systematic risk may be the only relevant investment decision since perfect diversification is costless. For the purposes of a practical theory of portfolio management, our definition of normative investment policy will include the level of portfolio diversification.

**Bache Halsey Stuart Shields,Inc., New York City. Earlier versions of this paper were presented to the Western Finance Association, June 1978, and to the Seminar on the Analysis of Security Prices, May 1976.*

[1]Three such studies are: Black, Jensen and Scholes [4]; Blume and Friend [7]; Fama and MacBeth [12].

[2]Recently, a number of studies have reported results which appear to contradict the Efficient Market Hypothesis. For a review of some of these studies see Ball [2].

One of the most serious defects of CAPM as a guide for practical investment management[3,4] is its single-period nature. Many portfolios under institutional management, including pension funds, endowment funds, and trust funds, have long-term investment objectives. What is required is an understanding of the multiperiod consequences of single-period investment policy decisions. Hakansson [14] has shown that mean-variance efficient portfolios may have a negative long-term expected geometric mean (growth rate, compound return) and, therefore, that continued reinvestment may lead to ruin with probability one. Although the return distributions in Hakansson's examples are not representative of "normal" portfolio return distributions, nevertheless, Hakansson's result is an important caveat to users of the mean-variance single-period models as a normative theory of investment. Under the stated assumptions, the results in Section III provide simple approximations for estimating the effect of (unvarying) single-period investment policy on the mean and variance of the portfolio's geometric mean return over the investment horizon.

The geometric mean often has significant explanatory power for many paradoxical results related to multiperiod investment. A simple example may be useful as an illustration (Block [5]): an investor invests 50 percent of his assets in risky securities in each time period; either his return matches the amount invested or it is lost; and there is a 50 percent chance of either outcome in each time period. Analysis reveals the paradoxical conclusion that this fair game leads to ruin with probability one. For $N \geq 2$, the expected geometric mean is negative and declines to the limit -13.4 percent, indicating the expectation of diminished wealth over time. As we will show, the expected geometric mean is, at least asymptotically, an estimate of the geometric mean of median terminal wealth. The expected geometric mean is an investment tool with the mathematical tractability and convenience of a mean that provides useful information concerning long-term median investment performance as a consequence of single-period investment decisions. Used with an awareness of its limitations, the statistical parameters of the *ex ante* geometric mean distribution can be a useful adjunct to the financial planning process.

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[3]Another fundamental critique of CAPM as a practical tool for portfolio management is Roll [30].

[4]The use of single-period capital market equilibrium models such as CAPM in a multiperiod framework is problematic. It implies either that intertemporal returns are not identically distributed or that the equilibrium market is equally weighted (Rosenberg and Ohlson [31]). What we will require is that the single-period return generation process is consistent with the Security Market Line (SML) of the CAPM.

A number of authors have examined properties of the parameters of the geometric mean distribution as an alternative portfolio selection criterion.[5] In particular, Hakansson [15] has proposed a mean-variance geometric mean (Hakansson) efficient frontier as a natural multiperiod generalization of the Markowitz [25] mean-variance efficient frontier. The Hakansson criterion, however, is often not consistent, even asymptotically, with expected utility maximization.[6] One basic objective of this paper is to examine the relationship of the parameters of the geometric mean distribution to the terminal wealth distribution and, thereby, clarify some strengths and limitations of a mean-variance geometric mean analysis of portfolio return.

Hakansson's criterion may be useful in many investment situations of practical importance. Hakansson [15] and Thorp [39] list a number of attractive investment objectives achievable by maximizing the limit of the expected geometric mean as $N \to \infty$ or, equivalently, expected log utility in each period. The portfolio growth rate is often an important part of stated investment objectives of professionally managed portfolios and is generally a critical consideration in the evaluation of *ex post* portfolio performance.

The mathematical results of this paper have direct application to Monte Carlo portfolio simulation studies which use the market line model for describing portfolio return in each period of an investment horizon (see e.g., Lorie and Hamilton [24, Ch. 15]). From our analysis we will be able to show the impact of capital market and investment policy assumptions on mutliperiod portfolio return underlying the simulation process. This can serve as a benchmark for evaluating how factors specific to a given situation affect portfolio returns and consequent funding behavior.

In Section II, we derive some theoretical results concerning the geometric mean distributional characteristics and relationship to the N-period terminal wealth ratio distribution. In Section III, using a statistically tractable approximation of the sample geometric mean, we derive the (approximate) relationship between single-period risk and diversification policies and N-period mean-variance geometric mean return. In Section IV, we display the expected geometric mean as a function of beta and discuss the application of our analysis to portfolio management and interpretation of long-term empirical data. In Section V, we provide a summary of our results.

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[5] These include: Kelly [21], Latane [22], Markowitz [25, Ch. 6], Breiman [9], Hakansson [15], Hakansson and Miller [17].

[6] See Samuelson [33], Merton and Samuelson [27], Samuelson and Merton [34], Hakansson and Miller [17], Markowitz [26], and Hakansson [16].

II. The Geometric Mean and N-Period Terminal Wealth Ratio: Some General Results

The definition of geometric mean return over N investment periods[7] is

$$G_N(R) = \sqrt[N]{(1 + r_1)\ (1 + r_2)\ldots(1 + r_N)} - 1 \tag{1}$$

where R will denote a vector of returns $r_1, r_2, \ldots, r_N$ in the N investment periods. We assume that returns $r_i > -1$, $i=1,\ldots, N$, are intertemporally independent and identically distributed. The quantity $1 + r_i$ will be referred to as the holding period return or wealth ratio in period i.

Property 1: The expected geometric mean is a decreasing function of the number of periods.[8]

To show this we write the expected geometric mean in the form

$$E(G_N(R)) = (E(1+r)^{\frac{1}{N}})^N - 1. \tag{2}$$

$E(G_N(R))$ in (2) defines an L_p norm and can be shown to be a monotone nonincreasing (generally decreasing) function of N (Thomas [39, p. 317]).

Property 2:

$$G_N(R) \xrightarrow{a.s.} e^{E(\ln(1+r))} - 1 \tag{3}$$

$$\text{as } N \longrightarrow \infty;$$

i.e., $G_N(R)$ converges almost surely (strong law of large numbers), to the constant $e^{E(\ln(1+r))}-1$. This is a standard result (e.g., Markowitz [25, Ch. 6]) and can be proven using the well-known almost sure convergence of $\ln(G_N(R)+1) \longrightarrow E(\ln(1+r))$ and the continuity of e^x.

Property 2 is a justification of the equivalence of maximizing the expected geometric mean with maximizing $E(\ln(1+r))$, when N is large (e.g., Markowitz [25, Ch. 6]; Samuelson [33]). In any real application, the limit is not reached. It is, therefore, of interest to know how a decision based on the limit criterion differs from the expected geometric mean criterion for given finite N. Hakansson [14, Sec. 11], using Monte Carlo simulation, examines

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[7] This definition presumes that cashflows are absent and all proceeds are reinvested in subsequent periods over the investment horizon.

[8] This result provides an alternative derivation and a reinterpretation of the downward bias properties of the geometric mean found in Blume [6].

special cases of the "intermediate run" problem. An important part of the analysis in this paper is to provide analytic estimates of statistical parameters and probabilities of the intermediate-run geometric mean distribution.

Define the parameters: $A=E(\ln(1+r))$, $B=V(\ln(1+r))$. It is well known that the distribution of the geometric mean is asymptotically lognormal with mean: $e^{A+B/(2N)}-1$; variance: $e^{2A+B/N}(e^{B/N}-1)$, (e.g., Aitchison and Brown [1, Ch 2]). It can also be shown, however, that

Property 3: $G_N(R)$ is approximately normally distributed with mean: e^A-1; variance: Be^{2A}/N for large N.

This result follows from Rao [29, pp. 319-320].

Some intuition concerning the distribution of the geometric mean for large N can be derived from examining the case when holding period returns are assumed to be lognormally distributed in each period. This assumption implies that the geometric mean is lognormally distributed for any value of N. This lognormal distribution, however, has statistical characteristics similar to that of a normal distribution when N is large; i.e., the mean approaches the median and the measures of departure from normality, the coefficients of skewness and kurtosis approach zero (Aitchison and Brown [1, pp. 8-9]). Ultimately, the geometric mean distribution converges to a point distribution.

As a consequence of the distributional properties of the geometric mean for large N, it follows that the mean and variance are asymptotically relevant parameters for describing the multiperiod portfolio return distribution. An important corollary is that the mean is a consistent estimator of the median of the geometric mean. The asymptotic distributional properties are also important for linking expected geometric mean return to the median of the terminal wealth distribution.

N-period terminal wealth, in units of initial wealth (the wealth ratio over N periods), can be written as: $W_N(R) = (1+r_1)(1+r_2)\ldots(1+r_N)$. Consequently N-period terminal wealth and the geometric mean are related according to $W_N(R) = (G_N(R)+1)^N$. Let W_p and G_p denote, respectively, the p^{th} quantiles in the terminal wealth and geometric mean distributions for some fixed investment horizon.

Property 4:

$$W_p = (G_p+1)^N. \tag{4}$$

Equation (4) expresses the fundamental relationship between the geometric mean and terminal wealth distributions. It can easily be derived from the fact that W_N is a monotone increasing function of G_N. In particular, (4) shows the direct relationship between the medians of the two distributions for any N. To the extent that the normal distribution describes the distribution of the geometric mean, the mean and variance provide, via (4), a useful description of the N-period terminal wealth distribution. Assuming asymptotic properties hold, the median of the geometric mean distribution is $e^{\hat{A}}-1$, which by (4), is the growth rate of median terminal wealth. $E(G_N(R))$ is, in the asymptotic normal case, equal to $e^{A}-1$ and, in the asymptotic lognormal case, equal to $e^{A+B/2N}-1$. Therefore, the expected geometric mean can be useful as an estimate of median investment performance over the investment horizon when N is sufficiently large. It should be noted, however, that a symmetric interval about the mean in the geometric mean distribution corresponds to an asymmetric interval about (approximately, for large N) the median of terminal wealth.

The asymptotic distribution of N-period terminal wealth is lognormal. This can easily be shown by applying the central limit theorem to the log of the definition of $W_N(R)$. This result implies that the distribution of single-period returns over increasingly long time periods will tend toward lognormality.

The basic asymmetry of the asymptotically increasingly right-skewed N-period terminal wealth distribution is not reflected in the asymptotically symmetric geometric mean distribution. This fact is one of the fundamental differences between the single-period and the multiperiod investment problem. Given the context of expected utility maximization over terminal wealth and possible high utility of the asymptotic right-skew of the terminal wealth distribution (Hakansson [16]), considerable care must, therefore, be exercised in inferring useful investment decisions from the geometric mean distribution.

From the definition of $W_N(R)$ it follows that

$$E(W_N(R)) = (1+\mu)^N \tag{5}$$

i.e., the mean of single-period returns (μ) is directly related to the mean of terminal wealth. In skewed distributions, the median is often the descriptive parameter of choice for central tendency. In concrete investment terms, the probability that any given investor will achieve the mean may be very small. In highly right-skewed terminal wealth distributions, median terminal wealth is an estimate of the investment performance experienced by a typical individual or institution over the investment horizon. The appropriateness of the mean-variance geometric mean criterion to multiperiod investors is, therefore, related to the appropriateness of portfolio objectives described in terms of median terminal wealth and risk measures consistent with asymmetric intervals according to (4) about the median, when N is large.

Given the validity of our theoretical assumptions, our analysis has some implications for the interpretation of geometric means computed from empirical studies of long-run rates of return. On the one hand, an average terminal wealth ratio may be computed and the results reported as a geometric mean. In this case, the geometric mean is simply a summary statistic of the average of the terminal wealth distribution. If, however, terminal wealth ratios are expressed as geometric means and an average geometric mean is computed, the result is a summary statistic (asymptotically) related to the growth rate of median terminal wealth.

III. Investment Policy and the Geometric Mean Mean-Variance Parameters

An approximation of the sample geometric mean return is used to compute the mean and variance of the geometric mean return distribution as a function of (fixed) single-period risk and diversification policy when the number of periods is finite. The effect of single-period investment policy on multiperiod return and terminal wealth is described and contrasted with the single-period case.

A number of approximations exist which express the sample geometric mean as a function of the moments of the returns (Young and Trent [42]). For purposes of mathematical tractability, simplicity of derived results, and consistency with the assumption of a mean-variance description of the single-period return distribution, the one we shall use is

$$G_N(R) \doteq \bar{r} - \frac{s^2(R)}{2} \tag{6}$$

where $\bar{r}$ is the average and $s^2(R)$ is the (biased) sample variance of the given returns.

Young and Trent [42] have shown that (6) can be a useful approximation of the sample geometric mean for monthly and, to a lesser extent, yearly historical capital market returns. A significant, but small, downward bias was observed. The validity of the approximation (6) is dependent on the adequacy of the mean-variance description of the single-period return distribution, the serial independence assumption, and the values of the mean-variance parameters. The approximation can break down badly in some investment settings of practical interest.

The single-period return generation process is assumed to be consistent with the SML of the Sharpe-Lintner CAPM; i.e.,

(7) $$\mu = E(R) = R_o + \beta(E(M)-R_o) = R_o+\beta\Delta$$

where

(8) $$\beta = COV(R,M)/\sigma_M^2.$$

Using a CAPM framework, the symbols in (7) and (8) are defined as follows: R is the total return on the capital asset or portfolio for the period; M is the return on the market portfolio (value weighted) of all assets taken together; R_o is the return on a riskless asset for the period;[9] COV(R,M) is the covariance between the asset or portfolio and the market portfolio; and σ_M^2 is the variance of return of the market portfolio. Using (8) and by definition of the correlation, ρ, the standard deviation of returns can be written as

(9) $$\sigma_R = \beta\sigma_M/\rho.$$

Equations (7) and (9) completely specify the single-period return-risk relationships for an asset or portfolio of assets for the assumed return generation process. For our purposes, the market environment is characterized by the capital market parameters: expected market return; standard deviation of market returns; and the risk-free rate.

To compute the variance of the geometric mean in terms of the mean and variance, we have assumed that the third central moment of single-period returns is equal to zero and the fourth central moment is equal to three times the square of the variance.

The expected value for the geometric mean return (6) can be computed using standard statistical techniques, e.g., Hogg and Craig [18]. It then follows from our assumptions that the expected N-period geometric mean is approximately equal to

(10) $$E(G_N(R)) = \mu-(1-1/N)\sigma^2/2.$$

Using equations (7) and (9), the expected value of $G_N(R)$ can be written in terms of the single-period investment policy and capital market parameters

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[9] It is a simple exercise to derive this section's results when R_o is a random variable which may or may not have a nonzero correlation with market return in each period.

(11) $$E(G_N(R)) \doteq R_o+\beta\Delta-(1-1/N)\beta^2\sigma_M^2/(2\rho^2).$$

Proceeding in the same way, we can derive the variance of $G_N(R)$ as follows (see Fisz [13, p. 369, 9.2 and p. 371, 9.17]):

(12) $$V(G_N(R)) \doteq (1+(1-1/N)\sigma^2/2)\sigma^2/N$$

which can be written as

(13) $$V(G_N(R)) \doteq (1+(1-1/N)\beta^2\sigma_M^2/(2\rho^2))\beta^2\sigma_M^2/(\rho^2 N).$$

The approximation (11) is an N-period expected geometric mean description of a return-generation process which is consistent in each period with the single-period SML.[10] Of particular interest is the role of total risk (9) implicit in (11) and the fact that increasing diversification increases the expected N-period geometric mean.

For N>1, the formula for the expected geometric mean (11) is a quadratic function of beta with a maximum value[11] at

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[10] The results of our analysis may be extended to include a multifactor return-generating process such as Ross [32]. In this case, the geometric mean parameters are functions of a vector of the coefficients of the systematic risk factors and the level of diversification.

[11] The validity of most of the results of this and the next section depends on the approximative power of (11). There are, however, important characteristics of the expected geometric mean that are not captured by this simple approximation. In the Appendix, two more theoretically accurate approximations are examined. The Appendix approximations show, among other things, that: (a) the critical beta may fail to exist if $\sigma_M<2\Delta$ or when N is small; and (b) the expected geometric mean may not be well approximated by a quadratic function of beta beyond the critical value. The increase in theoretical accuracy of the Appendix approximations results in an increase in the complexity of the solutions. In practical terms, the differences between the three solutions are often, though not always, small. In particular, the approximation of the critical beta (14) is often a significant underestimate; the approximations (3A) and (9A) are, generally, the estimates of choice. In order to examine the accuracy of the analytic approximations of this section, and of those in the Appendix, Monte Carlo simulations of multiperiod investment were performed over a 20-period investment horizon for various historical values of annual capital market parameters for values of beta from zero to two. Each period's return was generated using a (left) Truncated Normal Distribution derived from the IBM-SSP Gauss subroutine. The results showed that the approximations were, generally, reasonable estimates of the simulated geometric mean return distribution's statistical parameters. In the following, we shall assume that "typical" values of capital market parameters are used and that a normal range of the investment policy parameters of large stock portfolios is under consideration. Under these assumptions the approximations will provide a useful description and a simple understanding of the relationships between investment policy and capital market parameters and N-period mean-variance geometric mean.

(14) $$\beta_{C,N} \doteq \Delta\rho^2/((1-1/N)\sigma_M^2).$$

$\beta_{C,N}$ is the market risk policy which maximizes the expected geometric mean over N investment periods. For future convenience, we define $\beta_C = \lim_{N\to\infty} \beta_{C,N}$.

The value of the "critical" beta defined in (14) denotes not only an optimal expected geometric mean risk level, but also a maximum sensible level of market risk in the given market environment with respect to the growth rate of the asset or portfolio given a fixed level of diversification. Since the variance in (13) is an increasing function of market risk, increasing beta beyond the critical value will increase the variance while decreasing the expected value of the growth rate.

In terms of the terminal wealth distribution (assuming asymptotic properties hold for the finite N-period investment horizon), increasing beta beyond its critical value will: (a) decrease median terminal wealth; and (b) increase mean terminal wealth.[12] The many attractive multiperiod properties of the mean-variance geometric mean criterion result in part from the statistical properties of the median in skewed distributions. As Samuelson [33] points out, however, an individual with a Pascal utility function ($U(W)=W$) will not be impressed with the median's properties. Such considerations illustrate some of the subtleties and pitfalls of long-range investment planning; in the absence of a specific utility function, multiperiod investment policy decisions must be made with considerable care and attention to the valid needs, objectives, and resources of the individual or institution.

By definition, portfolios on the single-period capital market line are perfectly correlated with the market portfolio. An examination of (11) and (13) will show that for a given value of β, $E(G_N(R))$ increases and $V(G_N(R))$ decreases as $\rho \longrightarrow 1$. Hence, to the extent that our approximations are valid, the fixed investment policy N-period mean-variance geometric mean efficient frontier is composed of a subset of portfolios on the single-period capital market line.

IV. The Geometric Mean Criterion and Portfolio Management

The importance of a theoretical tool in a practical investment setting is related to the magnitude of the effects it predicts and to its relevance in normal investment decision making. By evaluating the expected geometric mean (11)

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[12] This result follows from (5) and (7).

for capital market parameters of historical[13] and current interest, it can be shown that critical betas may exist within the range of normal investment policy and that the multiperiod effects of single-period assumptions and investment policy on the growth rate of the portfolio can differ significantly from the simple linear model.

In Figure 1, which follows, a display of the market risk-geometric mean relationships given by equations (11) and (13) is shown for a portfolio with a correlation of 0.99 with the market. An expected yearly (single-period) market return of 15 percent, a standard deviation of 25 percent, and risk-free rate of 8 percent were assumed. The choice of annual data as the single-period and the values of the capital market parameters are for illustrative purposes only.[14] Alternative definitions of the single-period do not substantively alter many of the conclusions of this section.[15] The curves labeled 5th and 95th percentile are derived on the basis of the assumption that the geometric mean is normally distributed.

The expected geometric mean curves demonstrate that the substantial gains in wealth which will, on average, accrue to high-risk portfolios, tend to be achieved by an increasingly small group of investors. For the market assumptions of Figure 1, $\beta_C = 1.10$ (from (9a), $\beta_C = 1.41$). If we increase σ_M or decrease the risk premium (Δ), the critical beta will decrease; e.g., for the assumptions of Figure 1, if σ_M is increased to 30 percent, $\beta_C = 0.76$.

Let $\beta_C{}^*$ represent the fixed risk and diversification policy associated with the long-term critical beta (β_C) for single-period efficient portfolios (ρ=1). $\beta_C{}^*$ is the risk and diversification policy associated with maximizing the long-term expected geometric mean. Therefore, to the extent that our

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[13] Sources of historical capital market data include Ibbotson and Sinquefield [19] and Sharpe [37, Table 8.1].

[14] An important unresolved issue, of particular relevance for developing a positive theory of long-term capital market behavior, is the definition of an appropriate single-period. For this analysis the empirical issues concern the existence of time periods where assumptions of intertemporal independence and a single-period return generation process consistent with a single- (or multi-factor SML, are justifiable.

[15] Significant differences may occur over intermediate-length time periods. In particular, assume that a year is composed of 12 (monthly) single-periods for which the return generation assumptions hold. Under these conditions the annual return distribution may possess a critical beta.

ANNUAL GEOMETRIC MEAN: EXPECTED AND EXTREME VALUES
EXPECTED MARKET RETURN = 15 PCT
MARKET STD DEV = 25 PCT
RISK FREE RATE = 8 PCT
MARKET CORRELATION = 0.99

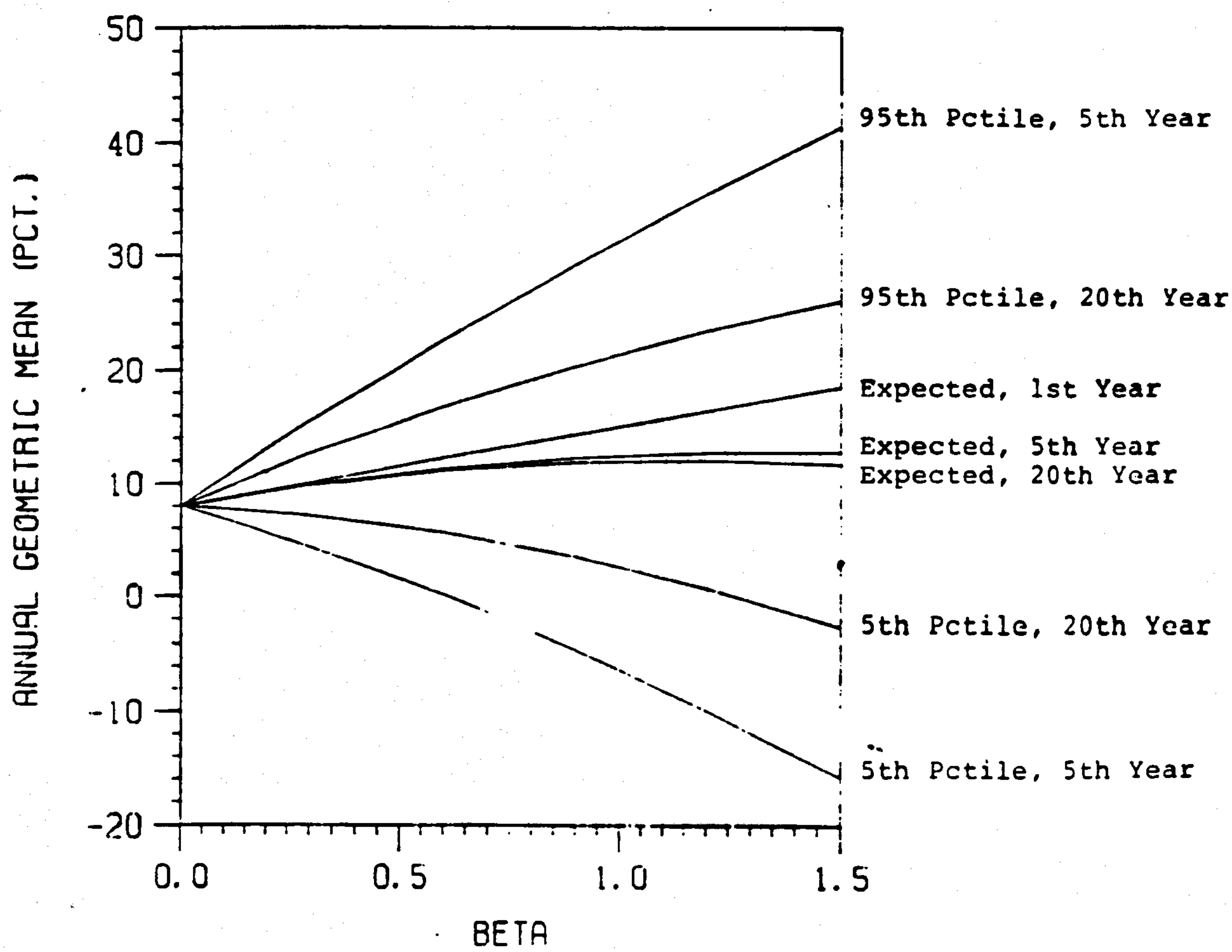

Figure 1

approximations are valid, any other "essentially different" (varying or unvarying) risk and diversification policy will almost surely have less growth rate and terminal wealth than the $\beta_C{}^*$ policy given a sufficiently long period of time (Thorp [40, pp. 252-254]). Evaluating $\lim_{N \to \infty} E(G_N(R))$ at $\beta_C{}^*$ for $\rho=1$ provides an estimate of the almost sure maximum growth rate (see (10A) in the Appendix). For the values of the parameters in Figure 1, when $\rho=1$, the almost sure maximum growth rate is approximately 12.7 percent.

Diversification takes on added importance with respect to multiperiod portfolio return. The diversification-expected geometric mean relationship makes it difficult for the typical portfolio manager with a less than perfectly diversified portfolio to beat a perfectly diversified benchmark for a given risk class. Over sufficiently long periods of time, mean wealth ratios over the N-period investment horizon will be the same for two portfolios with equal betas, but the median wealth ratio will be less for the less well diversified portfolio.

As a normative theory of investment, the problem we have described is concerned solely with managing multiperiod investment return. As a practical tool of portfolio management, it suffers from the omission of such critical considerations as accumulated wealth, consumption-investment decisions, and cashflow requirements in each period. Monte Carlo simulation has been proposed as a technique for modeling the financial operation of a fund over time in order to develop an appropriate long-term investment policy (Lorie and Hamilton [24, Ch. 15]). This technique, however, is heir to the Samuelson-Merton objections of the Hakansson criterion.

Except for two points on the Hakansson efficient frontier--zero variance and maximum expected geometric mean return--or when returns in each period are lognormally distributed (Hakansson and Miller [17]), constant (proportion) risk policies do not lead to Hakansson efficient portfolios. Dynamic programming computer solutions of Hakansson criterion optimal investment strategies in the simple one risky and one riskless asset case show that an optimal intertemporal investment strategy in beta may deviate substantially from a comparable fixed beta strategy over the investment horizon (Michaud and Monahan [28]).[16] Therefore, optimal intertemporal investment policy with respect to the Hakansson criterion is not, in general, constant over a multiperiod investment horizon.

Since the variance of the geometric mean is asymptotically zero and the

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[16] These computational solutions provide counter-examples to an assertion by Hakansson [16, p. 174, A5]) concerning the one risky and one riskless asset case.

maximum mean is a decreasing function of the number of periods, the fixed investment policy mean-variance geometric mean efficient frontier may provide a reasonable approximation of the Hakansson efficient frontier, especially for investors with indefinite or very long time horizons. While small differences in the growth rate may imply large differences in terminal wealth over long investment horizons, the benefits associated with Hakansson optimal solutions in comparison with fixed investment policies may be mitigated by the likely increase in transaction costs.

The theoretical results relating beta to long-term return and terminal wealth may be useful as a positive theory of long-term capital market behavior. The long-term return-beta data of Sharpe [38, p. 292] and Blume and Friend [8] is qualitatively consistent with the predicted relationship (11). We note, however, that the presence of a small, but significant, serial correlation in Blume and Friend's data does contradict our assumption of intertemporal independence. Also, the theoretical relationship (11) assumes a constant level of diversification while the empirical portfolio return distributions will exhibit a concave level of diversification as a function of beta with a maximum at or near one.

To some, these empirical results may appear anomalous, since, under our assumptions, average terminal wealth would appear to be a convex, not concave, function of beta. This paradox is explainable from considerations of random sampling in highly right-skewed distributions. Assuming asymptotic properties hold, we will show that the nature of the relationship between beta and the average terminal wealth ratio is dependent on the number of observations in the sample average.

Given intertemporal independence, the (theoretical) wealth ratio distribution is asymptotically lognormal, increasingly right-skewed, and such that the mode, median, and mean follow in the order given. Assuming asymptotic properties hold, one observation from a 40-year wealth ratio distribution is most likely to represent an estimate of the mode of the distribution. Under these conditions, the long-term data in Blume and Friend [8, Table 4] and Sharpe are far more likely to represent estimates near the median than the mean of terminal wealth. Again, assuming asymptotic properties hold, the distribution of an eight-sample average from a significantly right-skewed distribution is likely to be right-skewed as well. Superimposed upon the original wealth ratio distribution, the sample average distribution will have the same mean, but because of the central limit theorem, the mode and median will shift to the right towards the mean. An eight-sample average such as Blume and Friend [8, Table 1] will most likely represent an estimate of the mode of the sample average distribution, which in this case is a point between the mode and mean in the original distribution and may be near the median of the original distribution.

The foregoing analysis was not intended to provide corroboration of data with theory. Additional, more appropriate, empirical data will be required. Rather, its primary purpose was to demonstrate the statistical subtleties that may exist in the interpretation of long-term return data and to show that the existence of an empirical concave function of beta with the (average) terminal wealth ratio does not rule out the existence of a monotonic increasing relationship of risk with expected terminal wealth.

Probability estimates of geometric mean returns have often been derived using Monte Carlo simulation (e.g., Williamson [41, p. 83]) and have been considered of practical importance in long-range investment planning and in evaluating the investment implications of various actuarial return assumptions. Our analysis, however, lends itself to simple analytic estimates of geometric mean probabilities. When N is large, equations (10) and (12) or (11) and (13) may be used to form a standardized random variable which is approximately normally distributed.

V. Summary and Conclusions

An intertemporally independent and stationary return generation process consistent in each (discrete, uniform size) period with the SML of the CAPM, was assumed. Under these conditions, approximations of the statistical characteristics of the geometric mean distribution were derived that provide simple and convenient estimates of the multiperiod consequences of fixed single-period investment policy decisions on portfolio return. Within the range of validity of the approximations, it was shown that: (a) the expected geometric mean is approximately a quadratic function of beta; (b) a critical beta (generally) exists beyond which the expected growth rate diminishes while the variance increases; and (c) increasing diversification increases the expected growth rate while decreasing the variance.

The relationship of the geometric mean and terminal wealth distribution has been explored. Because of the asymptotic properties of the geometric mean distribution, median terminal wealth is approximately a concave function of beta when N is large.

The results of the theoretical analysis of the effect of beta and diversification on the N-period terminal wealth distribution may be useful as a positive theory of long-term capital market behavior. The assumptions used are implicit in many empirical tests of return-beta linearity. The theoretical results suggest that the long-term empirical relationship between beta and wealth ratios should be convex for the mean and concave for the median. The long-term wealth ratio data of Blume and Friend [8] and Sharpe [38] were examined and found consistent with our analysis. Important open empirical issues remain, however, including the definition of an appropriate single-period.

While the evaluation of the multiperiod consequences of fixed single-period investment policy parameters based on the statistical characteristics of the N-period terminal wealth or geometric mean return distribution may have intuitive appeal as a normative investment theory, it is, nevertheless, subject to serious reservations. The Samuelson-Merton objections to the geometric mean criterion are fundamentally concerned with the fallacy of attending to the statistical characteristics of the terminal wealth distribution as surrogates for the mean of the utility of terminal wealth distribution. The investment problem which we have considered is concerned solely with managing multiperiod portfolio return and ignores critical considerations in investment management such as accumulated wealth levels and cashflow requirements in each period. In addition, fixed investment policies are generally suboptimal for investors with a geometric mean portfolio selection criterion.

Nevertheless, the distribution of the portfolio growth rate is of interest in many investment situations of practical importance. An effort to clarify the long-term risk-return relationship has led to new tools for analyzing the multiperiod consequences of fixed single-period investment decisions. If it is used with an awareness of its limitations, many institutions and investors may be supplied with a useful benchmark for evaluating the impact of single-period investment policy parameters on their investment objectives.

For practical portfolio management, perhaps the most important aspects of this analysis are the awareness of the possible limitations of high-beta securities and portfolios, the critical role of market parameter assumptions, the role of diversification for investors with long-range investment objectives, and the limitations on growth rates of capital in financial markets.

APPENDIX

ALTERNATIVE APPROXIMATIONS OF THE GEOMETRIC MEAN

1. Using Taylor's formula for N variables, expanding (1) about the point $\underline{\mu}$, and ignoring terms of order three and higher or those with zero expectation, we derive the approximation

$$G_N(R) \doteq \bar{r} - s_o^2(1-1/N)/(2(1+\mu)) \text{ where } s_o^2 = \Sigma(r_i-\mu)^2/N.$$

Then

$$E(G_N(R)) \doteq R_o+\beta\Delta-(1-1/N)\ \beta^2\sigma_M^2\ /(2\rho^2(1+R_o+\beta\Delta)) \tag{1A}$$

$$V(G_N(R)) \doteq (1+(1-1/N)^2\beta^2\sigma_M^2/(2\rho^2(1+R_o+\beta\Delta)^2))\beta^2\sigma_M^2/(\rho^2 N) \tag{2A}$$

and, assuming $\Delta>0, N > 1$,

$$\beta_{C,N} \doteq (1+R_o)\ ((1-2\rho^2\Delta^2/(\sigma_M^2(1-1/N)))^{-1/2}-1)/\Delta. \tag{3A}$$

From an analysis of the estimates (1A)-(3A), it follows that: (a) the expected geometric mean as a function of beta may not be well approximated by a quadratic, particularly for betas larger than the critical value; (b) the variance of the geometric mean is dependent on single-period mean return; (c) a critical beta may fail to exist if $\sigma_M/\Delta<\sqrt{2}$ or for small N; and (d) the approximations for the critical beta and expected geometric mean are uniformly larger and the variance of the geometric mean uniformly smaller than their corresponding values in Section III. Note that the derivative of (1A) as a function of beta evaluated at $\beta=0$ is Δ. Therefore, assuming consideration of $\beta\geq 0$, the approximation (1A) implies that when the expected risk premium $\Delta\leq 0$, the efficient frontier is a point and the optimal (varying or unvarying) investment policy is not to put any money at risk.

2. An alternative approximation of $E(G_N(R))$ will be derived which has the correct asymptotic limit (3), and which provides more accurate estimates of the geometric mean characteristics when N is large. For r suitably constrained, we write the binominal series expansion

$$\begin{aligned}(1+r)^{\frac{1}{N}} &= 1+r/N-(1-1/N)r^2/(2N)+(1-1/N)(2-1/N)r^3/(2\cdot 3N)-\ldots \\ &= 1+(1/N)(r-(r^2/2)(1-1/N)+(r^3/3)(1-1/2N)(1-1/N)-\ldots)\end{aligned} \tag{4A}$$

$$\doteq 1+(\ln(1+r))/N+\phi(N)/N$$

where $\phi(N) \longrightarrow 0$ as $N \longrightarrow \infty$.

From (4A) and (2), when N is large, the expected geometric mean can be approximated by

$$E(G_N(R)) \doteq (1+E(\ln(1+r))/N)^N-1 \tag{5A}$$

which has the limiting value (3) as $N \longrightarrow \infty$.

Using the fact that

$$E(\ln(1+r)) \doteq \ln\,(1+\mu) - \sigma^2/(2(1+\mu)^2) \tag{6A}$$

we can approximate the expected geometric mean when N is large, in terms of the single-period mean-variance of returns

$$E(G_N(R)) \doteq (1+(\ln(1+\mu))/N-\sigma^2/(2N(1+\mu)^2))^N-1. \tag{7A}$$

From (3) and (6A) we may derive more accurate estimates of the long-term (expected) geometric mean, the critical beta (assuming $\Delta>0$), and the maximum long-term growth rate, E*, which is achievable with probability one, given the single-period market assumptions:

$$\lim_{N\to\infty} E(G_N(R)) \doteq (1+R_o+\beta\Delta)e^{-\beta^2\sigma_M^2/(2\rho^2(1+R_o+\beta\Delta)^2)}-1 \tag{8A}$$

$$\beta_c \doteq (1+R_o)\;(\sigma_m^2 - 2\Delta^2\rho^2 - \sigma_m\sqrt{\sigma_m^{\,2} - 4\Delta^2\rho^2})/(2\Delta^3\rho^2) \tag{9A}$$

$$E^* \doteq (1+R_o)\sigma_M(\sigma_M-\sqrt{\sigma_M^2-4\Delta^2})/(2\Delta^2)e^{-(\sigma_M-\sqrt{\sigma_M^2-4\Delta^2})^2/(8\Delta^2)}-1. \tag{10A}$$

Note that β_C in (9A) exists only if $\sigma_M \geq 2\Delta$ and is always, and often substantially, larger than the corresponding estimate in (14).

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