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Best rotated minimax approximation

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BEST ROTATED MINIMAX APPROXIMATION

by

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ABSTRACT

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(Order No.______)

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In this dissertation we consider the minimax approximation of functions $f(x) \in C[0, 1]$ rotated about the origin, and the characterization of the optimal rotation, α^*, of f in the sense of least minimax error over all possible rotations. The paper divides naturally into two sections: a) Existence, uniqueness, and characterization for unisolvent minimax approximation for each rotation α of f. These results are applications of Dunham (1967). b) Existence, non-uniqueness, and computation of α^*; derivation of necessary conditions for the minimax approximation at α^*, which are applications of Curtis and Powell (1966); and documentation of the effect of rotation on the minimax error by means of computed results. The parameter space for the minimax error for the rotations of f is $[0, \pi]$. Hence, a parameter search for the computation of α^* is feasible. However, we also designed an algorithm for computing α^* by means of an iterated linear programming approach due to Esch and Eastman (1968). In most of the cases we have examined, the error of approximation at α^* has n+2, rather than the necessary n+1, equioscillating extrema. However, we show by example that the existence of n+2 equioscillating extrema at α^* is neither necessary, nor sufficient, nor locally sufficient. Some typical results are given which show that a quadratic minimax approximation at α^*

frequently has smaller error than a cubic minimax approximation. If the minimax error at α^* is comparable to the minimax error of a higher degree approximation, then the minimax approximation at α^* is, in general, a better curve fit, since it has fewer changes in direction. A fairly complete theory for straight line approximation is presented. Finally, some theoretical results on the form of the error function at a minimax approximation at α^* are given, which are extensions of some results due to Tornheim (1950).

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CHAPTER I

INTRODUCTION

The central theme of this paper is the minimax approximation of a function f rotated by an angle α about the origin and the characterization of the angle of rotation of least minimax error. The resulting curve in the X, Y coordinate system will be referred to as an "α-rotation" of f.

Let f(x) be an arbitrary given continuous real-valued function on $[0,1]$. Let $G = \{g(A,x)\}$ be a given set of admissible approximations and let $\|\cdot\|$ be the uniform or L_∞ norm on $[0,1]$. Then a minimax or best approximation (B. A.) to f is a $g^* = g(A^*, x)$ belonging to G such that

$$\|f(t) - g^*(t)\| \le \|f(t) - g(t)\| = \sup_{t \in [0,1]} |f(t) - g(t)|$$

for all $g \in G$. Here A stands for the real parameters $a_1, a_2, \ldots, a_n$ of the approximating function g.

The class of approximations which we shall consider is the unisolvent class of some fixed degree n.

Definition 1.1: The class G_n of n-parameter continuous functions is said to be unisolvent of degree n on $[a,b]$, if for any distinct points $x_1, \ldots, x_n$ of $[a,b]$ and any real numbers $y_1, \ldots, y_n$ there is a unique $g(A,x) \in G_n$ such that the equations

$$g(A, x_i) = y_i \quad, \qquad i = 1, \ldots, n$$

are satisfied.

This definition implies that the difference of two distinct $g_1, g_2 \in G_n$ can have at most n-1 zeros.

We will wish to consider two special cases of unisolvent functions of degree n: The class L_n of Chebychev or generalized polynomial approximations (Rice, 1964, p. 55) of the form: $\sum_{i=1}^{n} a_i \phi_i(x)$; and P_{n-1}, the polynomial class of degree n-1 or less. The class P_{n-1} has the additional property of being unisolvent over any closed interval.

Let f be parametrized by $t \epsilon [0, 1]$ and let $y_\alpha(t)$ denote the point f(t) rotated by an angle α. We denote the abcissa of the point $y_\alpha(t)$ in the X, Y coordinate system, $x_\alpha(t)$. The error function of an α-rotation of f by $g \epsilon G_n$ is:

$$e_\alpha(t) = y_\alpha(t) - g(A, x_\alpha(t)) \quad .$$

$e_\alpha(t)$ is evidently periodic in α with period 2π.

We define $e^*(\alpha)$ as the minimax error for an α-rotation of f, and $e^*(\alpha^*)$ as the infimum of $e^*(\alpha)$ for all rotations $\alpha \epsilon [0, 2\pi]$. A best rotated approximation (B.R.A.) is a $g^*_{\alpha^*} \epsilon G_n$ such that $g^*_{\alpha^*}$ is a B.A. for an α^*-rotation of f and the uniform error of approximation is $e^*(\alpha^*)$.

We will be concerned with seeking the solution to two classes of problems:

1) Existence of B.A. for any α-rotation of f and characterization and uniqueness of B.A. in terms of the sign of the error at extrema of the error function $e_\alpha(t)$ for the approximating class G_n.

2) The existence, uniqueness, and computation of α^*, the characterization of B.R.A., for the Chebychev and polynomial approximating classes, and the documentation of the effect of rotation on the minimax error.

For many approximating classes of interest; e.g., the polynomial class P_{n-1}, the minimax error is invariant with respect to a translation of the function, but the orientation of f with respect to rotation can significantly effect the minimax error. If our purpose is to compute values of f accurately on some closed interval, then a minimax approximation (without rotation) is ideal. However, if we wished to fair points $f(x_i)$, $i = 1, \ldots, N$, given as design specifications for some physical surface, it is often the case that there is no a priori reason to prefer one orientation over another. If accuracy of approximation is the only factor to be considered, then we can dispense with questions of optimal orientations of the data by allowing sufficiently high degree polynomial approximations. But for many applications, particularly in computer-aided design, a curve fairing process requires the approximation to be relatively free of extraneous changes in direction or "wiggles". In this regard, a polynomial B.R.A. of degree n would, in general, have one less "wiggle" than a B.A. of degree n+1. The need for "wiggle-free" approximations has been, in part, responsible for proscribing the application of minimax theory to automated curve fitting. The angle α^* defines an optimal curve fitting procedure with respect to the uniform norm and the given approximating class. A B.R.A. is rotation-independent in the sense that neither the approximation $g^*_{\alpha^*}$ nor the error $e^*(\alpha^*)$ depend on the given orientation of the function.

The problem of characterizing α^* has been extensively studied in the case of straight line approximation for the discrete ℓ_2 norm. The best approximating line for α^* is called the orthogonal regression line (Linnik, 1961). Roos (1937) has given a solution for a best approximating straight line in the ℓ_2 norm which is invariant to rotation, translation, and linear stretching. Michaud (1969) has given conditions for α^* for straight line approximation in the L_2 norm.

In Chapter II we review the existence, uniqueness, and characterization theorems of minimax approximation, examine the α-rotation error $e_\alpha(t)$ in detail and relate the minimax theorems to the set R, of α-rotations, such that the curve remains a function after rotation. We also use some examples to demonstrate that neither uniqueness nor full characterization are general properties of α-rotation minimax approximation. These examples also serve to motivate the definitions and results of the following chapter.

In Chapter III we redefine the α-rotation minimax approximation problem as the simultaneous approximation of upper and lower semicontinuous envelopes of a uniformly bounded map and apply the results of Dunham (1967) to prove existence, conditions for uniqueness, and necessary and sufficient conditions for the α-rotation minimax approximation of f.

In Chapter IV we define the best rotated approximation problem. We then show that for every continuous function on $[0, 1]$ or any subset thereof, possesses a B.R.A. for the Chebychev class of approximations. The proof is a consequence of our result that $e^*(\alpha)$ is a continuous function of α. We prove that α^* is not in general unique. We then apply some results due to Curtis and Powell (1966) to obtain necessity conditions for the best rotated approximation.

In Chapter V we present an algorithm for computing α^* and the polynomial B.R.A. We then give some numerical results obtained using the algorithm and a parameter search technique. For most of the cases examined, the error of the B.R.A. had one more equioscillating extrema than necessary, coinciding with those instances for which the effect of the rotation parameter on the minimax error was most significant. However, we give examples which demonstrate that the error function of a B.R.A. cannot, in general, be characterized by n+2 equioscillating extrema.

In Chapter VI we consider best rotated straight line approximation in detail. In Chapter VII we present extensions of some results due to Tornheim (1950) for characterizing the error function at a B.R.A.

CHAPTER II

THE ERROR FUNCTION OF AN α-ROTATION OF f

2.1 THE CLASSICAL MINIMAX THEOREMS

Results of the classical minimax theory to which we will refer are the following (Rice, 1964):

Theorem 2.1 (Existence): If f is a continuous real-valued function on $[0, 1]$ then a minimax or best approximation $g^* \in G_n$ exists.

Theorem 2.2 (Uniqueness): The minimax approximation g^* of theorem 2.1 is unique.

Theorem 2.3 (Characterization): A function g^* is a best approximation if and only if the error function, $e(x) = f(x) - g^*(x)$, has $n + 1$ equioscillating extrema; i.e., there exist $n + 1$ distinct "critical points," x_i, $0 \le x_1 < x_2 < \ldots < x_{n+1} \le 1$, such that

$$e(x_i) = (-1)^i \lambda, \quad i = 1, \ldots, n+1, \quad \text{where } |\lambda| = ||e(x)||.$$

If $e(x)$ satisfies the conditions of theorem 2.3 then $e(x)$ is said to have "full characterization."

2.2 THE ERROR FUNCTION $e_\alpha(t)$

Let $f(t)$ be a continuous real-valued function on $[0, 1]$. We will use $t \in [0, 1]$ to parametrize this function after rotation. Before rotation the function is defined by $x = t$, $y = f(t)$. After rotation by an angle α (which we will measure positive clockwise), the curve is given by the parametric equations

$$\begin{aligned} x &= x_\alpha(t) = t\cos\alpha + f(t)\sin\alpha \\ y &= y_\alpha(t) = -t\sin\alpha + f(t)\cos\alpha \end{aligned} \qquad (2.1)$$

where t ranges from 0 to 1. We will refer to the curve (2.1) as the result of an "α-rotation" of f. If $x_\alpha(t)$ is monotone, the y given by (2.1) is a function of x; otherwise it is not; i.e., the curve has become multi-valued.

The range of abscissa values of the α-rotated f is the range of the continuous function $x_\alpha(t)$, which we will denote as $[a,b]$. Thus if we wish to exploit unisolvence, the class of approximations must be unisolvent over the interval $[a,b]$.

Now consider an approximation $g(A,x) \in G_n$ to the α-rotated f. Parametrized by t, the error

$$e_\alpha(t) = y_\alpha(t) - g(A, x_\alpha(t)),\ t \in [0,1] \tag{2.2}$$

is a continuous function in t and periodic with period 2π in α. We note for future reference that $|e_\alpha(t)|$ and hence $\|e_\alpha(t)\|$ is periodic in α with period π for polynomial approximation. If $G_n = L_n$ then (2.2) becomes

$$e_\alpha(t) = y_\alpha(t) - \sum_{i=1}^{n} a_i\varphi_i(x_\alpha(t)),\ t \in [0,1] \tag{2.3}$$

and if $G_n = P_{n-1}$ then (2.2) is of the form

$$e_\alpha(t) = y_\alpha(t) - \sum_{i=0}^{n-1} a_1(x_\alpha(t))^i,\ t \in [0,1]. \tag{2.4}$$

2.3 THE SET OF ROTATIONS R

Definition 2.4: $R = \{\alpha \mid$ the α-rotation of f is a function$\}$.

Proposition 2.5: Let G_n be a unisolvent class of functions of degree n defined over a range $[a,b]$ which is sufficiently large to include the range of values of $x_\alpha(t)$ for all $\alpha \in [0,2\pi]$. Then $\alpha \in R$ if and only if the set of functions G_n is unisolvent in t.

Proof: If $\alpha \in R$, the mapping $t \to x_\alpha(t)$ is one to one, and $x_\alpha(t)$ is a monotone function of t. Given distinct $t_1, \ldots, t_n \in [0, 1]$, and any real numbers $y_1, \ldots, y_n$ then $x_\alpha(t_1), \ldots, x_\alpha(t_n)$ are distinct points of $[a, b]$ and hence we can solve $g(A, x_\alpha(t_i)) = y_i$, $i = 1, \ldots, n$ uniquely.

If $\alpha \notin R$, there exists $t_1, t_2 \in [0, 1], t_1 \neq t_2$, such that $x_\alpha(t_1) = x_\alpha(t_2)$; we cannot then have

$$g(A, x_\alpha(t_1)) = y_1 \text{ and } g(A, x_\alpha(t_2)) = y_2.$$

Theorem 2.6: If $\alpha \in R$, then α-rotation minimax approximation exists, is unique, and is characterized by $n+1$ equioscillating extrema in t.

Proof: By theorems 2.1, 2.2, 2.3 and proposition 2.5.

2.4 THE α-ROTATION OF f AS A UNIFORMLY BOUNDED MAP

The purpose of this section is to anticipate some results and to motivate the approach taken in the following chapter on α-rotation minimax approximation.

Uniqueness of approximation is not a general property of α-rotation minimax approximation. Let f(x) be the continuous function of Fig. 2.1 and let Fig. 2.2 be its rotation by 90°.

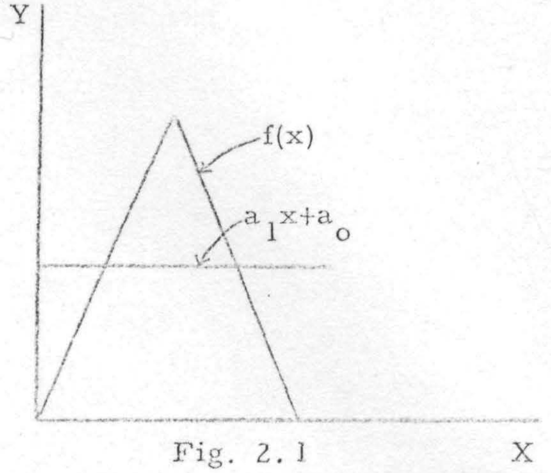


Fig. 2.1

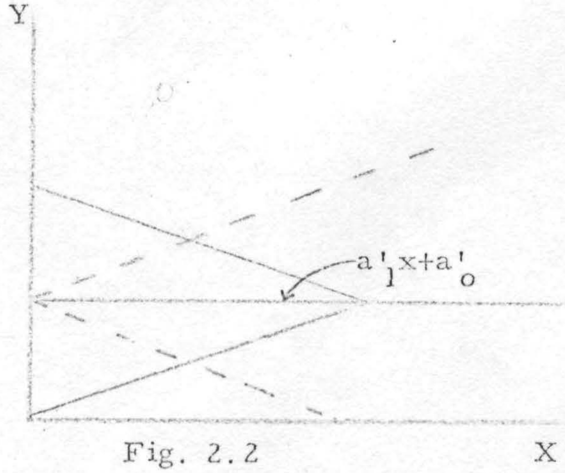


Fig. 2.2

If our set of approximations is the class of straight lines, then by theorem 2.2, a_1x+a_o in Fig. 2.1 is unique. The approximation $a_1'x+a_o'$ in Fig. 2.2 is clearly not unique since any straight line within the indicated pencil of lines would minimize the maximum error.

It is not possible in general to characterize a best approximation of an α-rotation of f by equioscillating extrema of the error function $e_\alpha(t)$ as in theorem 2.3. The following example serves to illustrate this point:

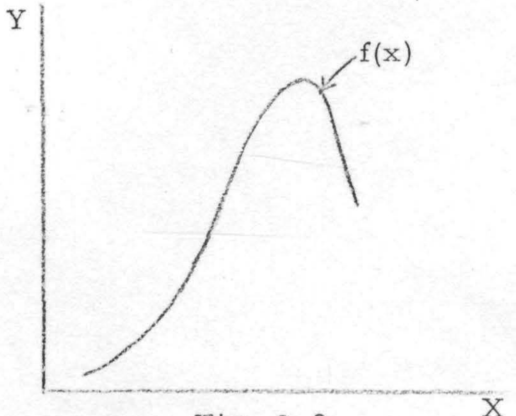


Fig. 2.3

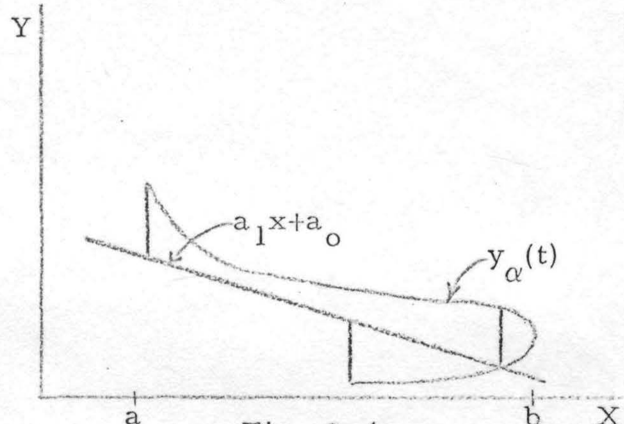


Fig. 2.4

Fig. 2.3 is the given continuous function f(x) is to be approximated by a straight line (so that G_n is a unisolvent class of degree two). In Fig. 2.4 f has been rotated by an angle α and the resulting curve is best approximated by the straight line a_1x+a_0. Fig. 2.5 is the resulting error function $e_\alpha(t)$ of Fig. 2.4 graphed as a function of t.

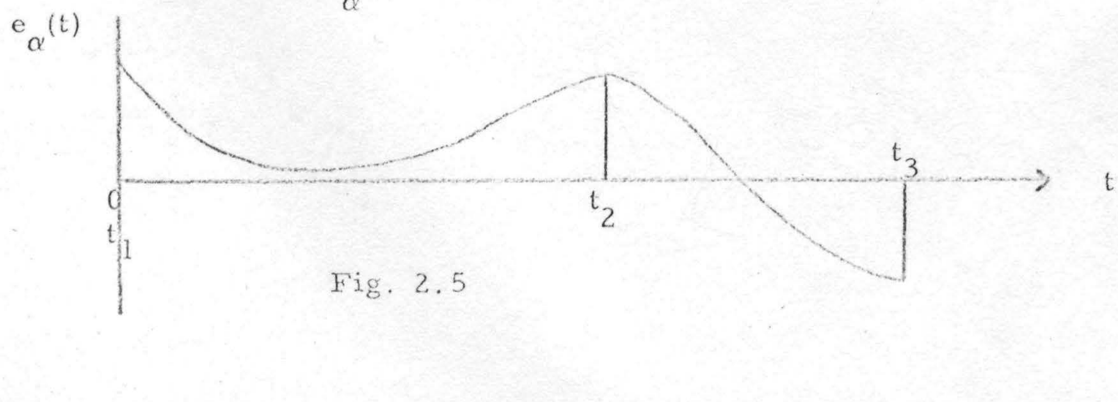


Fig. 2.5

Although $e_\alpha(t)$ has three $(n+1)$ extrema, they fail to oscillate.

However, if we plot the error with abscissa x instead of t, then we obtain Fig. 2.6.

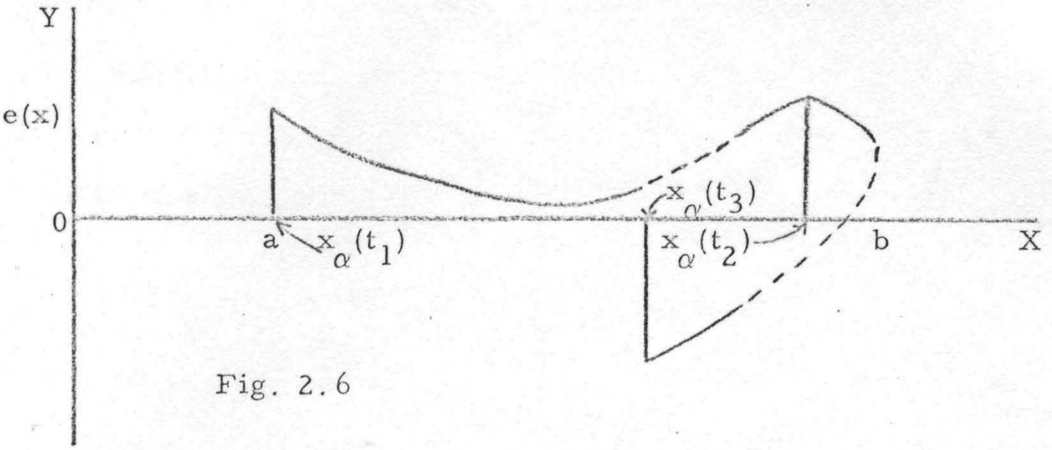


Fig. 2.6

We define the error e(x) by taking, in each region where the curve is multivalued, the branch lying farthest from $y = 0$ (as shown by the solid portions of the curve in Fig. 2.6). We obtain e(x) as an, at most, bi-valued map on $[a, b]$ which has discontinuities. However we thereby regain the $n+1$ equioscillation property, with critical points $a \le x_\alpha(t_1) < x_\alpha(t_3) < x_\alpha(t_2) \le b$ as seen on Fig. 2.6.

The redefinition of the error illustrated in Fig. 2.6 has become the crucial observation leading to the characterization and uniqueness theorems of the following chapter. We note that the equioscillation property in x is an easily verifiable condition on the extrema of $e_\alpha(t)$. We need only compute $x_\alpha(t_i)$, renumber the t_i's such that $x_\alpha(t_{i_1}) < x_\alpha(t_{i_2}) < x_\alpha(t_{i_3})$ and examine the signs of $e_\alpha(t)$ at $t_{i_1}, t_{i_2}, t_{i_3}$ to determine whether the error $e_\alpha(t_{i_j})$ does equioscillate.

The error function e(x) of the curve given in Fig. 2.4 is simple to compute since the curve is at most bi-valued as a mapping in x. However this is not in general the case. Such examples as $\sin(n\pi x)$ and $x\sin(1/x)$ when rotated by 90° can be multivalued or infinitely valued at a given value of x. We have the following obvious result:

Proposition 2.7: If f(x) is a continuous real-valued function on $[0, 1]$, then the α-rotation of f, defined as a mapping from $x \in [a, b]$, is a uniformly bounded map.

CHAPTER III

THE MINIMAX APPROXIMATION OF AN α-ROTATION OF f

3.1 INTRODUCTION

In this chapter we show that the uniform approximation of an α-rotation of f in x can be defined in terms of the simultaneous uniform approximation of an upper and lower semicontinuous function. Our constructed approximation problem is exactly that considered in Dunham (1967). Dunham's results give a solution to the uniqueness and characterization of a minimax approximation of an α-rotation of f. However, these conditions depend on properties at the extrema of the constructed approximation in x. We bypass this limitation by relating the extrema of $e_\alpha(t)$ with the extrema of the constructed semicontinuous approximation and proving the equality of the norm of the errors. We obtain, as a result, for each α-rotation of f: 1) existence of a minimax approximation, 2) necessary and sufficient conditions for minimax approximation in terms of the signs of the error at extrema of $e_\alpha(t)$, and 3) conditions for uniqueness of minimax approximation in terms of the extrema of $e_\alpha(t)$.

3.2 THE UPPER AND LOWER ENVELOPES OF AN α-ROTATION OF f

In the construction which follows we implicitly assume a given $\alpha \in [0,2\pi]$. We will not need to consider whether the α-rotated f is many valued in x or single valued, and hence the following analysis is intended to apply to any α-rotation of f.

Let $\alpha \in [0,2\pi]$, $[a,b]$ be the range of $x_\alpha(t)$, $t \in [0,1]$ and

$$T_x = \{t \mid x_\alpha(t) = x, \quad x \in [a,b]\}. \tag{3.1}$$

Then

$$f_+(x) = \sup_{t \in T_x} y_\alpha(t) \qquad f_-(x) = \inf_{t \in T_x} y_\alpha(t) \tag{3.2}$$

$$\tilde{e}_+(A,x) = f_+(x) - g(A,x) \qquad \tilde{e}_-(A,x) = f_-(x) - g(A,x) \tag{3.3}$$

$$\tilde{e}(A) = \max \{ ||\tilde{e}_+(A,x)|| \tilde{e}_-(A,x)|| \} . \tag{3.4}$$

Let

$$f^+(x) = \inf_{\delta > 0} \sup_{|u-x|<\delta} f_+(u) \qquad f^-(x) = \sup_{\delta > 0} \inf_{|u-x|<\delta} f_-(u) \tag{3.5}$$

By definition, f^+ and f^- are the upper and lower envelopes of $y_\alpha(t)$ and hence are upper and lower semicontinuous functions of x. From (3.1) and (3.5) it follows that

$$f^-(x) \le f_-(x) \le f_+(x) \le f^+(x) \tag{3.6}$$

for all $x \in [a, b]$.

We define the corresponding error functions of f^+ and f^- as:

$$e_+(A,x) = f^+(x) - g(A,x) \qquad e_-(A,x) = f^-(x) - g(A,x) \tag{3.7}$$

$$e(A) = \max \{ ||e_+(A,x)||, \quad ||e_-(A,x)|| \} . \tag{3.8}$$

Since f^+ and f^- are respectively upper and lower semicontinuous functions of x, it follows that $e_+(x)$ and $e_-(x)$ are respectively upper and lower semicontinuous functions of x for continuous approximations $g(A,x)$. Since an upper semicontinuous function assumes it maximum value on a closed interval, and a lower semicontinuous function assumes it minimum on a closed interval (Royden, 1963, p. 40), then $e_+(x)$ assumes its maximum and $e_-(x)$ its minimum on a closed interval.

3.3 THE DUNHAM THEOREMS

The approximation problem defined by $e(A)$ of (3.8) is exactly that of Dunham (1967). Before we consider Dunham's results, we make the following definitions:

Definition 3.1: An extremum point of the approximation $g(A,x)$ is a point $x_o \in [a,b]$ such that $f^+(x_o) - g(A,x_o)$ or $f^-(x_o) - g(A,x_o)$ is equal to $\pm e(A)$.

Definition 3.2: If x_o is an extremum point of $g(A,x)$ such that

$$f^+(x_o) - g(A,x_o) = -(f^-(x_o) - g(A,x_o)) = e(A)$$

then x_o is said to be a straddle point.

Definition 3.3: Extremum points of $g(A,x)$ which are not straddle points are termed alternation points.

The results of Dunham (1967) which we will refer to are the following:

Theorem 3.4(Dunham): A minimax or best approximation $g^*(x) = g(A^*,x) \in G_n$ exists which minimizes the uniform error to f^+ and f^- for all $g \in G_n$.

Theorem 3.5(Dunham): $g^*(x)$ is a B.A. if and only if $g^*(x)$ has a straddle point or n+1 alternation points oscillating in sign.

Theorem 3.6(Dunham): If g^* has n+1 alternation points oscillating in sign then g^* is unique.

3.4 RESULTS OF MINIMAX APPROXIMATION OF AN α-ROTATION OF f

Our goal is now to relate α-rotation approximation in x, about which we know a good deal, to approximation in $e_\alpha(t)$.

The following result due to Diaz and McLaughlin (1969) will be useful.

Theorem 3.7: $e(A) = \tilde{e}(A)$.

<u>Lemma 3.8</u>: $\widetilde{e}(A) = \|e_\alpha(t)\|$.

We will denote $\|e_\alpha(t)\|$ as $e_\alpha(A)$.

Proof: We will assume positive extrema throughout the proof.

It is evident that $\sup|\widetilde{e}_+(x)| \geq e_\alpha(A)$ since $y_\alpha(t) \leq f_+(x)$ for all $t \in T_x$.

Assume $\sup|\widetilde{e}_+(x)| > e_\alpha(A)$, and let t_1 be an extremum of $e_\alpha(t)$. Then by hypothesis, $e_\alpha(A) = y_\alpha(t_1) - g(A, x_\alpha(t_1))$ and

$$y_\alpha(t_1) < f_+(x), \quad x = x_\alpha(t_1).$$

But $y_\alpha(t_1) - g(A, x_\alpha(t_1)) \geq y_\alpha(t) - g(A, x_\alpha(t))$ for all $t \in [0,1]$

$$\geq y_\alpha(t) - g(A, x_\alpha(t)) \text{ for } t \in T_x.$$

Hence $y_\alpha(t_1) - g(A, x_\alpha(t_1)) \geq \sup y_\alpha(t) - g(A, x)$

$$= f_+(x) - g(A, x) .$$

But this is a contradiction.

<u>Theorem 3.9</u>: $e_\alpha(A) = e(A)$.

Proof: By lemma 3.8 and theorem 3.7.

<u>Theorem 3.10</u>: $t_o \in [0,1]$ is a point of extrema of $e_\alpha(t)$ if and only if $x_\alpha(t_o)$ is an extremum of $e(A)$.

Proof: We will prove the theorem only for positive extrema of $e_\alpha(t)$ and extrema of $e_+(x)$.

Evidently, if t_o is an extremum of $e_\alpha(t)$ then $x_\alpha(t_o)$ is an extremum of $e_+(x)$.

Now, let x_o be an extremum of $e_+(x)$ and let

$$T = \{t \mid e_\alpha(t) = \max e_\alpha(t)\} .$$

We assume there exists x_o such that for all $t \in T$, $x_\alpha(t) \neq x_o$; i.e., we are assuming that it is possible to create extrema by constructing semi-continuous envelopes of an α-rotation of f. Then

$$\underset{t \in T}{e_\alpha(t)} > \underset{t \notin T}{e_\alpha(t)} \geq \underset{t \in T_{x_o}}{e_\alpha(t)}$$

where $T_{x_o} = \{t \mid x_\alpha(t) = x_o\}$.

Since $f^+(x_o) = \inf_{\delta > 0} \sup_{|x-x_o|<\delta} f_+(x)$, where $f_+(x_o) = \max_{t \in T_{x_o}} y_\alpha(t)$ and T_{x_o} closed by continuity of $x_\alpha(t)$, then $f_+(x_o) < f^+(x_o)$.

Hence there exists $\delta > 0$ such that for $x \in Z = \{x \mid |x-x_o| < \delta\}$, $Z_T = \{t \mid x_\alpha(t) = x, x \in Z\}$, for all $t \in Z_T$, $t \notin T$. Otherwise there exists an infinite sequence of $\{t_i\} \in T$ such that $e_\alpha(t_i) \geq e_+(x)$, $x \in Z$ which implies that there exists a limit at x_o and $x_\alpha(t') = x_o$, $e_\alpha(t') \geq e_+(x_o)$ by continuity of $e_\alpha(t)$.

Therefore $\underset{t \notin T}{e_\alpha(t)} \geq \underset{t \in Z_T}{e_\alpha(t)}$, which implies

$$\underset{t \in T}{e_\alpha(t)} > \underset{t \notin T}{e_\alpha(t)} \geq \sup_{t \in Z_T} e_\alpha(t) \geq \inf_{\delta > 0} \sup_{t \in Z_T} e_\alpha(t) = f^+(x_o) - g(A, x_o).$$

But this contradicts the fact that x_o is an extremum and the max of $e_\alpha(t)$ is equal to that of $e_+(x)$. This concludes the proof.

The essential implication of theorem 3.10 is that no extrema are created in constructing the upper and lower envelopes of an α-rotation of f in x. However it is possible that the relationship between extrema in $e_\alpha(t)$ and $e_+(x)$ or $e_-(x)$ is not one to one. The following corollary provides the solution to this question.

Corollary 3.11: i) If x_o is an alternation point of the approximation then there exists a unique $t_o \in T_{x_o}$ such that t_o is an extremum of $e_\alpha(t)$.

ii) If x_o is a straddle point of the approximation then there exists exactly two points $t_1, t_2 \in T_{x_o}$, $t_1 \neq t_2$, such that t_1 and t_2 are extrema of $e_\alpha(t)$.

Proof: Since an α-rotated continuous function is a Jordan arc, given any $x \in [a,b]$, if $t_1, t_2 \in T_x$, $t_1 \neq t_2$, then $y_\alpha(t_1) \neq y_\alpha(t_2)$.

Corollary 3.11 implies that if $g(A,x)$ has no straddle points then there are the same number of extrema of $e_\alpha(t)$ as alternation points of $g(A,x)$.

We now state existence, characterization, and uniqueness results for α-rotation minimax approximation.

Theorem 3.12: (Existence): The minimax approximation $g^* \in G_n$, of the α-rotation of a continuous function, exists.

Theorem 3.13 (Characterization): $g^* = g(A^*, x_\alpha(t)) \in G_n$ is a minimax approximation to an α-rotation of f if and only if

a) there exist extrema t_1, t_2 of $e_\alpha(t)$, $t_1 \neq t_2$ such that $x_\alpha(t_1) = x_\alpha(t_2)$ and $e_\alpha(t_1) = -e_\alpha(t_2)$ or

b) there exists n+1 extrema $t_1, \ldots, t_{n+1}$ of $e_\alpha(t)$ such that, for some reordering t_{i_j}

$$x_\alpha(t_{i_1}) < x_\alpha(t_{i_2}) < \ldots < x_\alpha(t_{i_{n+1}}) \text{ and } e_\alpha(t_{i_j}) = (-1)^j(\pm \|e_\alpha(t)\|).$$

Proof: By theorem 3.5 and corollary 3.11.

Theorem 3.14 (Uniqueness): If g^* has property b) of theorem

3.13 then g^* is the unique minimax approximation.

Proof: Corollary 3.11 and theorem 3.6.

CHAPTER IV

BEST ROTATED APPROXIMATION: THEORETICAL RESULTS

4.1 DEFINITIONS

In this chapter we shall be primarily concerned with the Chebychev and polynomial classes of approximations and hence with the error functions (2.3) and (2.4). However, we make the following definitions for the unisolvent class G_n.

Definition 4.1: Given a unisolvent class of functions G_n, a continuous function f on [0, 1], and a given rotation α, then $e^*(\alpha)$ is the error of the α-rotation minimax approximation; i.e.,

$$e^*(\alpha) = \sup_{t\in[0,1]} |y_\alpha(t) - g(A^*, x_\alpha(t))| = \|y_\alpha(t) - g(A^*, x_\alpha(t))\| \quad .$$

By the results of the previous chapter, g^* exists for each α. In this chapter we will be concerned with optimizing α; i.e., with finding α^* for which $e^*(\alpha)$ is least.

Definition 4.2: A best rotated approximation (B.R.A.) is a $g^*_{\alpha^*} \in G_n$ such that $g^*_{\alpha^*}$ is a best approximation for the α^*-rotation of f such that

$$e^*(\alpha^*) \le e^*(\alpha) \qquad \text{for } \alpha \in [0, 2\pi] \quad .$$

$e^*(\alpha)$ is a bounded periodic function in α with period 2π (period π for polynomial approximation). We shall now show that it is continuous for $g \in L_n$.

4.2 EXISTENCE OF BEST ROTATED APPROXIMATION

Lemma 4.3: If $f(x) \ne a_1 x + a_0$, then the length of the interval $[a_\alpha, b_\alpha]$, the range of $x_\alpha(t)$, is never zero, for all $\alpha \in [0, \pi]$.

Proof: Let $M(\alpha) = b_\alpha - a_\alpha$, be the length of the interval of the range of

$x_\alpha(t)$. Let $u = \inf_\alpha M(\alpha)$. $M(\alpha)$ is continuous and bounded on $[0,\pi]$ and hence achieves its infimum. If $u = 0$, then there exists α' such that

$$x_{\alpha'}(t) = t\cos\alpha' + f(t)\sin\alpha' = c \quad .$$

Now $\sin\alpha = 0$ when $\alpha = 0$ or π and hence when $M(\alpha)$ is not zero. Thus at α',

$$f(t) = -c/\sin\alpha' + t\cot\alpha' \quad .$$

But this case is ruled out by hypothesis. Therefore $u \neq 0$.

Theorem 4.4: If $f(x) \neq a_1x + a_0$ then $e^*(\alpha)$ is a continuous real-valued function of α, where $G_n = L_n$.

Proof: The proof is indirect. Suppose $e^*(\alpha)$ is not continuous; then there exists $\alpha_0 \in [0, 2\pi]$ such that $e^*(\alpha)$ is not continuous at α_0. Then there exists $\epsilon > 0$ such that in every δ-neighborhood of α_0 there exists α such that $|e^*(\alpha) - e^*(\alpha_0)| \geq \epsilon$.

We discuss separately the cases A and B.

Case A: $|e^*(\alpha) - e^*(\alpha_0)| = e^*(\alpha) - e^*(\alpha_0)$

$$= \|y_\alpha(t) - \Sigma\, a_k^\alpha \varphi_k(x_\alpha(t))\| - \|y_{\alpha_0}(t) - \Sigma\, a_k^{\alpha_0} \varphi_k(x_{\alpha_0}(t))\|$$

where a_k^α denotes the kth coefficient of the Chebychev best approximation of degree n for the α-rotation of f.

Since $\|y_\alpha(t) - \Sigma\, a_k^\alpha \varphi_k(x_\alpha(t))\| \leq \|y_\alpha(t) - \Sigma\, a_k^{\alpha_0} \varphi_k(x_\alpha(t))\|$ then $e^*(\alpha) - e^*(\alpha_0)$

$$\leq \|y_\alpha(t) - \Sigma\, a_k^{\alpha_0} \varphi_k(x_\alpha(t))\| - \|y_{\alpha_0}(t) - \Sigma\, a_k^{\alpha_0} \varphi_k(x_{\alpha_0}(t))\|$$

$$\leq \|y_\alpha(t) - y_{\alpha_0}(t)\| + \|\Sigma\, a_k^{\alpha_0}(\varphi_k(x_\alpha(t)) - \varphi_k(x_{\alpha_0}(t))\|$$

$$\le \|y_{\alpha}(t) - y_{\alpha_0}(t)\| + \Sigma |a_k^{\alpha_0}| \, \|\varphi_k(x_{\alpha}(t)) - \varphi_k(x_{\alpha_0}(t))\| \quad . \tag{4.1}$$

Since

$$\|\Sigma a_k^{\alpha_0} \varphi_k(x_{\alpha_0}(t))\| \le 2\|y_{\alpha_0}(t)\|$$

the coefficients of $\Sigma a_k^{\alpha_0} \varphi_k(x_{\alpha}(t))$ are bounded (Rice, 1964, pp. 24-25). For sufficiently small $\delta > 0$, since $y_{\alpha}(t)$ and $x_{\alpha}(t)$ are continuous functions of α, and the φ_k's are continuous functions, (4.1) can be made less than ϵ. This is a contradiction.

Case B: From case A we can immediately conclude a similar result to (4.1); i.e.,

$$e^*(\alpha_0) - e^*(\alpha) \le \|y_{\alpha_0}(t) - y_{\alpha}(t)\| + \Sigma |a_k^{\alpha}| \, \|\varphi_k(x_{\alpha}(t)) - \varphi_k(x_{\alpha_0}(t))\| . \tag{4.2}$$

Since for all α, $\|y_{\alpha}(t)\|$ is bounded, then there exists a constant N such that

$$\|\Sigma a_k^{\alpha} \varphi_k(x_{\alpha}(t))\| \le 2\,\|y_{\alpha}(t)\| \le N, \text{ for all } \alpha \in [0, \pi] \quad . \tag{4.3}$$

Let $\Phi(t, a_k) = \Sigma a_k^{\alpha} \varphi_k(x_{\alpha}(t))$

be a best approximation of $y_{\alpha}(t)$. Hence the $\{a_k^{\alpha}\}$ satisfy (4.3). If the $\{a_k^{\alpha}\}$ are unbounded, then there exists a K and a sequence $\alpha_1, \ldots, \alpha_j, \ldots$ such that $a_K^{\alpha_j}$ is unbounded; i.e., $|a_K^{\alpha_j}| \to \infty$, $j \to \infty$.

We can then choose a subsequence of α_j such that

$$\max_k |a_k^{\alpha_j}| = |a_K^{\alpha_j}| \tag{4.4}$$

Finally, since $\alpha \epsilon [0,2\pi]$, we choose a subsequence of α_j satisfying (4.4) such that $\alpha_j \to \alpha'$.

Now consider the sequence

$$\psi_j(t, b_k) = \Sigma (a_k^{\alpha_j}/a_K^{\alpha_j}) \varphi_k(x_\alpha(t)) = \Phi(t, a_k^{\alpha_j})/a_K^{\alpha_j} \to 0, \quad j \to \infty \quad . \tag{4.5}$$

Since the magnitudes of $|a_k^{\alpha_j}/a_K^{\alpha_j}|$ are all bounded by unity, we can extract a subsequence converging to b_k', and since (4.5) is a continuous function defined on a compact set then

$$\varphi_K(x_{\alpha'}(t)) + \sum_{\substack{k=0 \\ k \neq K}}^{n} b_k' \varphi_k(x_{\alpha'}(t)) = 0 \quad . \tag{4.6}$$

Equation (4.6) is defined on some non-zero interval by lemma 4.3. But this contradicts the unisolvence of (4.6). Hence a_k^α must be bounded for all α.

Therefore, for sufficiently small $\delta > 0$, (4.2) can be made less than ϵ, which is a contradiction.

<u>Proposition 4.5:</u> $e^*(\alpha)$ is a bounded continuous real-valued function periodic with period 2π, provided $f(x) \neq ax + b$, f continuous on $[0, 1]$.

<u>Corollary 4.6</u> (Existence): The optimal orientation α^* and the best rotated approximation $g_{\alpha^*}^* \epsilon L_n$ exists for continuous f defined on $[0, 1]$, provided $f(x) \neq ax + b$.

<u>Corollary 4.7</u> (Existence): If one of the φ_k's of the L_n approximating class is a non-zero constant, then α^* and $g_{\alpha^*}^*$ exists for all continuous f(x) defined on $[0, 1]$.

Proof: If $f(x) = ax + b$, there exists α^* such that $e^*(\alpha^*) = 0$ (see corollary 6.5).

From corollary 4.7 we can conclude that polynomial B.R.A. exists for all

$f \epsilon C[0,1]$. We note that if the functions φ_k, $k = 1, \ldots, n$, are linearly independent, the proof of theorem 4.4 is unchanged and hence the results, proposition 4.5 and corollaries 4.6 and 4.7 follow.

Corollary 4.8: If f is defined on a finite point set or any subset $\{t_i\}$ of $[0,1]$ such that the cardinality of the set $\{x_\alpha(t_i)\}$ is greater than or equal to n for all $\alpha \epsilon [0, 2\pi]$, then $e^*(\alpha)$ is continuous on $[0, 2\pi]$.

Proof: The condition on the cardinality of the set $\{x_\alpha(t_i)\}$ guarantees that there will be enough points at any α, and hence at α' of the proof of theorem 4.4, so that the contradiction following from equation (4.6) holds.

Corollary 4.9 (Existence, discrete case): If the cardinality of the set $\{x_\alpha(t_i)\}$ is greater than or equal to n for all $\alpha \epsilon [0, \pi]$, then α^* and $g^*_{\alpha^*} \epsilon L_n$ exists for f defined on any subset of $[0,1]$.

In the computation of α^* and $g^*_{\alpha^*}$, we will often find it desirable to replace the interval $[0,1]$ by a finite point set and seek an approximation which is optimum on that set. The following result due to Cheney (1966, p. 86) relates the continuous and discrete α-rotation minimax error. We will need to establish some notation. Let X_α be the range of $x_\alpha(t)$ and Y_α be the subset of points $x_\alpha(t_i) \leq X_\alpha$ for $\alpha \epsilon [0, \pi]$. We define

$$|Y_\alpha| = \max_{x \epsilon X_\alpha} \inf_{y \epsilon Y_\alpha} |x - y| \quad .$$

Theorem 4.10: $e^*(\alpha)|_{Y_\alpha} \to e^*(\alpha)|_{X_\alpha}$ as $|Y_\alpha| \to 0$. Evidently $e^*(\alpha)|_{Y_\alpha} \leq e^*(\alpha)|_{X_\alpha}$. Let α^*_d and α^*_c denote respectively the optimal rotation for f defined over the point set $\{t_i\}$ and $[0,1]$. Then $e^*(\alpha^*_d) \leq e^*(\alpha^*_c)$ and $e^*(\alpha^*_d) \to e^*(\alpha^*_c)$ as $\sup_\alpha |Y_\alpha| \to 0$. If α^*_c is unique, then $\alpha^*_d \to \alpha^*_c$ as $\sup_\alpha |Y_\alpha| \to 0$.

4.3 THE NON-UNIQUENESS OF α^*

We will prove by constructing an example that α^* need not be unique for continuous functions defined on [0, 1] for the P_o class of approximations. We will need a result from Chapter VI which says that a best rotated constant approximation must have three equioscillating extrema or a straddle point. By examining Fig. 2.1, we see that there exist essentially three candidates for B.R.A., those given in Fig. 2.1 and 2.2, and in Fig. 4.1 below.

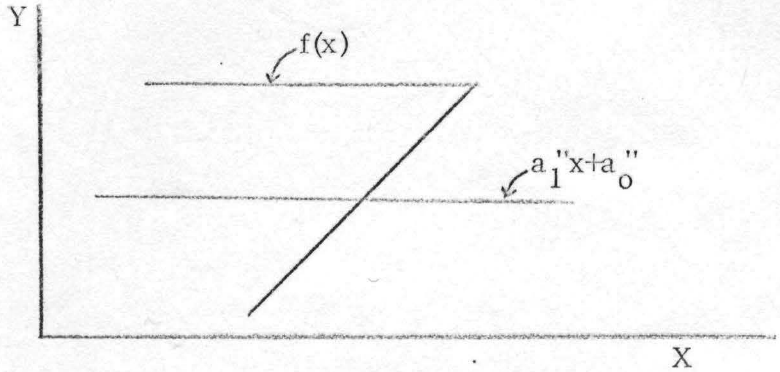


Fig. 4.1

From purely geometric considerations, the approximation of Fig. 4.1 has smaller minimax error than Fig. 2.2. If we set $h = \frac{\sqrt{3}}{2}$ the resulting minimax errors at $\alpha = 0$ or Fig. 2.1 and α of Fig. 4.1 are equal. Hence α^* is non-unique. We can also conclude that if $h < \frac{\sqrt{3}}{2}$ then $\alpha^* = 0 \in R$, and if $h > \frac{\sqrt{3}}{2}$ then $\alpha^* \notin R$.

4.4 CHARACTERIZATION OF BEST ROTATED APPROXIMATION

4.4.1 The goal of this section is to apply some results in the theory of non-linear minimax approximation due to Curtis and Powell (1966) to characterizing Chebychev and polynomial best rotated approximations where we assume $f \in C^1[0, 1]$. For the L_n class of approximations we will need to assume that the φ_k's have continuous first derivatives on [a, b]. $\Phi(t, \lambda^*) = \Phi(t, \lambda_1^*, \ldots, \lambda_{n+1}^*)$ is a minimax approximation to f(t) if the parameters λ_i^* are such that

$$\max_{t \in [0,1]} |f(t) - \Phi(t, \lambda_1^*, \ldots, \lambda_{n+1}^*)| \tag{4.7}$$

is minimized.

The minimax approximation of $e_\alpha(t)$ is not in the form (4.7) of the Curtis and Powell (C-P) problem. However, $e_\alpha(t)$ can easily be rewritten to fit the C-P paradigm.

$$e_\alpha(t) = f(t) - \Phi(t,\lambda) = f(t) - (f(t) - y_\alpha(t) + \Sigma a_k \varphi_k(x_\alpha(t))) \qquad t \in [0,1] \tag{4.8}$$

where the C-P approximating function is

$$\Phi(t,\lambda) = f(t) - y_\alpha(t) + \Sigma a_k \varphi_k(x_\alpha(t)) \qquad t \in [0,1] \tag{4.9}$$

and the C-P parameters are

$$\lambda_1 = a_1,\ \lambda_2 = a_2, \ldots, \lambda_n = a_n, \lambda_{n+1} = \alpha \quad .$$

Proposition 4.11: Let $\Phi(t,\lambda)$ be the approximating function (4.9). Let $\Phi(t,\lambda^*)$ be the minimax approximation defined by (4.7). Then a C-P minimax approximation is a best rotated approximation where $\lambda_i^* = a_i^*$ and $\lambda_{n+1}^* = \alpha^*$.

Proof: We denote the minimax error of (4.7) as $E(\lambda^*)$.

Evidently $E(\lambda^*) \le e^*(\alpha^*)$.

Given $\lambda_{n+1}^* = \alpha$, if $\lambda_i^* \ne a_i^*$ then $E(\lambda^*)$ is not minimax. If $\lambda_{n+1}^* \ne \alpha^*$ then by definition 4.2, $E(\lambda^*)$ is not minimax.

Definition 4.12: Let r equal the number of extrema of $e_{\alpha^*}^*(t)$ where $\Phi(t,\lambda^*) = \Phi(t, a_1^*, a_2^*, \ldots, a_n^*, \alpha^*)$. We define an r x n+1 matrix whose elements are given by

$$D_{ij} = \left.\frac{\partial \Phi(t_i,\lambda)}{\partial \lambda j}\right|_{\lambda=\lambda^*} \quad . \tag{4.10}$$

If h^* is the B.R.A. error then a_i is defined as the sign of the error $e_{\alpha^*}^*(t)$ at the extrema t_i

$$e_{\alpha^*}^*(t_i) = s_i\, k^*, \qquad i = 1, \ldots, r.$$

In the event that $r = n + 2$, there are $n + 2$ square matrices of order $n + 1$, denoted by $\Delta_1, \ldots, \Delta_r$, where Δ_k is the matrix obtained by deleting the kth row of the matrix D. ρ_k is reserved for the determinant of Δ_k multiplied by $(-1)^k$.

<u>Theorem 4.13</u> (Curtis and Powell): At λ^* the rank of D is less than r.

<u>Theorem 4.14</u> (Curtis and Powell): At λ^*, if $r = n + 2$, the signs of $s_1, \ldots, s_r$ are all the same as or all opposite to the signs of $\rho_1, \ldots, \rho_r$.

An examination of the proof of the Curtis and Powell theorems reveals that the C-P conditions are necessary at a relative minimum and hence must be satisfied at every relative minimum. For our purposes, the C-P theorems are tests, on the basis of which consideration can be narrowed to those approximations which satisfy the conditions.

By (4.10) the C-P matrix is:

$$D = \begin{pmatrix} \varphi_1(x_\alpha(t_1))\,\varphi_2(x_\alpha(t_1)) \ldots & \varphi_n(x_\alpha(t_1)) & y_\alpha(t_1)\,(\Sigma a_k \varphi_k'(x_\alpha(t_1)) + x_\alpha(t_1)) \\ \varphi_1(x_\alpha(t_2))\,\varphi_2(x_\alpha(t_2)) \ldots & \varphi_n(x_\alpha(t_2)) & y_\alpha(t_2)\,(\Sigma a_k \varphi_k'(x_\alpha(t_2)) + x_\alpha(t_2)) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \varphi_1(x_\alpha(t_r))\,\varphi_2(x_\alpha(t_r)) \ldots & \varphi_n(x_\alpha(t_r)) & y_\alpha(t_r)\,(\Sigma a_k \varphi_k'(x_\alpha(t_r)) + x_\alpha(t_r)) \end{pmatrix}$$

Since the crucial property of the C-P matrix with respect to the Curtis and Powell theorems is its rank, then D can be simplified to

$$D = \begin{pmatrix} \varphi_1(x_\alpha(t_1)) \ldots & \varphi_n(x_\alpha(t_1)) & y_\alpha(t_1)\,(\Sigma a_k \varphi_k'(x_\alpha(t_1))) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \varphi_1(x_\alpha(t_r)) \ldots & \varphi_n(x_\alpha(t_r)) & y_\alpha(t_r)\,(\Sigma a_k \varphi_k'(x_\alpha(t_r))) \end{pmatrix} \qquad (4.11)$$

At λ^*, $e^*_{\alpha^*}(t)$ has characterization according to theorem 3.13 and hence $e^*_{\alpha^*}(t)$ has either a straddle point extrema or at least n + 1 extrema in $t(r \geq n+1)$.

4.4.2 CASE: BEST ROTATED APPROXIMATION HAS A STRADDLE POINT EXTREMA

Theorem 4.15: If there exists one straddle point and r = n + 1 extrema, then a necessary condition for L_n B.R.A. is that the derivative of the approximation in x_α is zero at the straddle point.

Proof: By theorem 4.13, the rank of D must be less than n + 1. We assume the straddle point occurs at t_1 and t_2; i.e., $x_\alpha(t_1) = x_\alpha(t_2)$.

$$D = \begin{pmatrix} \varphi_1(x_\alpha(t_1)) \dots \varphi_n(x_\alpha(t_1)) & y_\alpha(t_1) & (\Sigma a_k \varphi_k'(x_\alpha(t_1)) \\ \varphi_1(x_\alpha(t_1)) \dots \varphi_n(x_\alpha(t_1)) & y_\alpha(t_2) & (\Sigma a_k \varphi_k'(x_\alpha(t_1)) \\ \varphi_1(x_\alpha(t_3)) \dots \varphi_n(x_\alpha(t_3)) & y_\alpha(t_3) & (\Sigma a_k \varphi_k'(x_\alpha(t_3)) \\ \vdots & & \\ \varphi_1(x_\alpha(t_{n+1}) \dots \varphi_n(x_\alpha(t_{n+1})) & y_\alpha(t_{n+1}) & (\Sigma a_k \varphi_k'(x_\alpha(t_{n+1})) \end{pmatrix} \quad (4.12)$$

which implies

$$|D| = (y_\alpha(t_1) - y_\alpha(t_2))\,(\Sigma a_k \varphi_k'(x_\alpha(t_1)) \begin{vmatrix} \varphi_1(x_\alpha(t_1)) \dots & \varphi_n(x_\alpha(t_1)) \\ \varphi_1(x_\alpha(t_3)) \dots & \varphi_n(x_\alpha(t_3)) \\ \varphi_1(x_\alpha(t_{n+1})) \dots & \varphi_n(x_\alpha(t_{n+1})) \end{vmatrix} \quad (4.13)$$

The third factor is non-zero since the set of functions $\{\varphi_k\}$ satisfies the Haar condition (Rice, 1964, p. 91). The first factor is never zero by the fact that the α-rotation of a continuous function is a Jordan arc. Hence the second term must be zero at a best rotated approximation. This is the derivative of the approximation evaluated at the straddle point.

If we apply theorem 4.15 to straight line approximation, we conclude that the slope at a B.R.A. must be zero. In chapter VI we will verify this result.

Theorem 4.16: If there exists $r = n + 1$ extrema then a necessary condition for L_n B.R.A. is the existence of two or more straddle points.

Proof: If there exists one straddle point, then (4.13) is valid. If there exists an additional straddle point, then the third factor of (4.13) must necessarily be zero.

4.4.3 CASE: BEST ROTATED APPROXIMATION HAS NO STRADDLE POINT EXTREMA

In this section we will be concerned with polynomial approximation. The C-P matrix for $G_n = P_{n-1}$ is:

$$D = \begin{pmatrix} 1 & x_\alpha(t_1) \dots & x_\alpha^{n-1}(t_1) & y_\alpha(t_1) & (\sum_{k=0}^{n-1} ka_k x_\alpha^{k-1}(t_1)) \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & x_\alpha(t_r) \dots & x_\alpha^{n-1}(t_r) & y_\alpha(t_r) & (\sum_{k=0}^{n-1} ka_k x_\alpha^{k-1}(t_r)) \end{pmatrix} \qquad (4.14)$$

By theorem 3.13 there exist $t_1, \dots, t_{n+1} \in [0,1]$ such that $x_\alpha(t_1) < \dots < x_\alpha(t_{n+1})$ and

$$\pm(-\epsilon)^i = y_\alpha(t_i) - \sum_{k=0}^{n-1} a_k x_\alpha^k(t_i) \quad , \quad i = 1, \dots, n+1 \qquad (4.14)$$

Then we can solve for ϵ, a_k by the linear system

$$\begin{pmatrix} 1 & x_\alpha^{n-1}(t_1) & \dots & x_\alpha(t_1) & 1 \\ -1 & x_\alpha^{n-1}(t_2) & \dots & x_\alpha(t_2) & 1 \\ 1 & & & & \\ \cdot & & & & \\ \cdot & & & & \\ (-1)^n & x_\alpha^{n-1}(t_{n+1}) & \dots & x_\alpha(t_{n+1}) & 1 \end{pmatrix} \begin{pmatrix} \epsilon \\ a_{n-1} \\ a_{n-2} \\ \cdot \\ \cdot \\ a_o \end{pmatrix} = \begin{pmatrix} y_\alpha(t_1) \\ y_\alpha(t_2) \\ \cdot \\ \cdot \\ \cdot \\ y_\alpha(t_{n+1}) \end{pmatrix} \qquad (4.15)$$

For polynomial approximation, the matrix D can be interpreted as a divided difference.

We denote $[x\, x_o \ldots x_n;\, f]$ the nth divided difference of a continuous function $f(x)$ on $[0,1]$ (Davis (1963) and Milne (1949)) where

$$[x\, x_o \ldots x_n;\, f] = \frac{\begin{vmatrix} 1 & x & x^2 & \ldots & x^n & f(x) \\ 1 & x_o & x_o^2 & \ldots & x_o^n & f(x_o) \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & x_n & x_n^2 & \ldots & x_n^n & f(x_n) \end{vmatrix}}{\begin{vmatrix} 1 & x & x^2 & \ldots & x_n^n & x^{n+1} \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & x_n & x_n^2 & \ldots & x_n^n & x_n^{n+1} \end{vmatrix}} \tag{4.16}$$

When $x \neq x_o, \ldots x_n$, the denominator of (4.16) is a non-zero Vandermonde determinant.

If $p_n(x)$ is the interpolating nth degree polynomial to the values of $f(x)$ at x_k, $f(x_k) = p_n(x_k)$, $k = 0, \ldots, n$, and $f \in C^{(n+1)}[0,1]$, then from interpolation theory, $f(x) = p_n(x) + R(x)\,\Pi\,(x - x_i)$ where

$$R(x) = [x\, x_o \ldots x_n;\, f] = \frac{f^{(n+1)}(s)}{(n+1)!}\ , \quad s \in (0,1)\ . \tag{4.17}$$

Consider the function $k_\alpha(t) = y_\alpha(t)\,\left(\sum_{k=0}^{n-1} ka_k\, x_\alpha^{k-1}(t)\right)$. (4.18)

When $\alpha \in R$, $k_\alpha \in C^{n+1}[0,1]$, provided $f \in C^{n+1}[0,1]$.

From (4.16) and (4.17) we define the divided difference of $k_\alpha(t)$ as $[t\, t_o \ldots t_n;\, k_\alpha]$ from which we conclude that the determinant of D is equal to

$$|D| = \frac{V(t)\, k_\alpha^{(n+1)}(s)}{(n+1)!}\ , \quad s \in (0,1)\ , \tag{4.18}$$

where $V(t)$ is a non-zero Vandermonde determinant and

$$k_\alpha^{(n+1)}(s) = \left.\frac{d^{(n+1)}k_\alpha}{dx_\alpha^{(n+1)}}\right|_s$$

Theorem 4.17: If $f \in C^{(n)}[0,1]$, $\alpha^* \in R$, and $k_{\alpha^*}^{(n)}(t) \underset{(<)}{>} 0$, $t \in (0,1)$, then $e_{\alpha^*}^*(t)$ must have at least n + 2 extrema.

Proof: From the condition that $\alpha^* \in R$, n + 1 equioscillating extrema are necessary. But by the condition on $k_{\alpha^*}^{(n)}$, if there exists only n + 1 extrema, the rank of the matrix D is n + 1 and therefore the approximation cannot be a B.R.A. by theorem 4.13.

Theorem 4.18: If $f \in C^{(n)}[0,1]$, $\alpha^* \in R$, $k_{\alpha^*}^{(n)}(t) \underset{(<)}{>}$, 0 for $t \in (0,1)$, and $e_{\alpha^*}^*(t)$ has exactly n + 2 extrema, then the extrema must equioscillate.

Proof: The n + 2 extrema must satisfy the conditions of theorem 4.14. But by the condition on $k_{\alpha^*}^{(n)}$ all the Δ_k determinants are positive (or negative) which implies that the extrema must oscillate in sign.

4.4.3.1 P_2 APPROXIMATION

Proposition 4.19: If the quadratic B.R.A. has no straddle points and if $f^{(2)}(x) \underset{(<)}{>} 0$, $x \in (0,1)$, then $a_2^* \neq 0$.

Proof: By (4.15) we can solve for a_2^*; i.e.,

$$a_2^* = \begin{vmatrix} 1 & y_\alpha(t_1) & x_\alpha(t_1) & 1 \\ -1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ -1 & y_\alpha(t_4) & x_\alpha(t_4) & 1 \end{vmatrix} / \Delta \qquad (4.19)$$

The denominator of (4.19), Δ, is not zero provided straddle point extrema do not exist. Using (2.1) the numerator of (4.19) can be reduced to

$$\begin{vmatrix} 1 & t_1 & f(t_1) & 1 \\ -1 & t_2 & f(t_2) & 1 \\ +1 & t_3 & f(t_3) & 1 \\ -1 & t_4 & f(t_4) & 1 \end{vmatrix} \quad . \tag{4.20}$$

Let $k(t) = c_1 + c_2 t + c_3 f(t)$ for real c_i and $t \in [0, 1]$. $k(t)$ can have at most two zeros for all $t \in [0, 1]$; for, if not, $k'(t)$ can have two or more zeros, $t \in (0, 1)$, which implies $h''(t)$ can have one or more zeros $t \in (0, 1)$. But this is impossible. Hence the last three columns of (4.20) are linearly independent. The first column is independent of the last three columns since $k(t)$ can change sign at most two times, $t \in [0, 1]$.

Proposition 4.20: If the quadratic B.R.A. has $a_2^* = 0$ and $\epsilon^* \neq 0$, then $a_1^* = 0$.

Proof: We can write the determinant of the matrix (4.14) as

$$|D| = a_1^* \begin{vmatrix} 1 & x_\alpha(t_1) & x_\alpha^2(t_1) & y_\alpha(t_1) \\ \cdot & & & \\ \cdot & & & \\ 1 & x_\alpha(t_4) & x_\alpha^2(t_4) & y_\alpha(t_4) \end{vmatrix} + 2a_2^* \begin{vmatrix} 1 & x_\alpha(t_1) & x_\alpha^2(t_1) & y_\alpha(t_1)x_\alpha(t_1) \\ \cdot & & & \\ \cdot & & & \\ 1 & x_\alpha(t_4) & x_\alpha^2(t_4) & y_\alpha(t_4)x_\alpha(t_4) \end{vmatrix}$$

which, by our assumptions and (4.15), reduces to

$$|D| = a_1^* \epsilon^* \Delta \quad .$$

By hypothesis, ϵ^* and $\Delta \neq 0$. Hence, at a B.R.A., $a_1^* = 0$.

4.5 SOME LIMITING PROPERTIES OF α^*

It may occur that, for a given degree Chebychev or polynomial approximation, a B.R.A. is useless or uninteresting as in the case of Fig. 4.1. However, we can consider this result to be due to the fact that our class of approximations was not of sufficiently high degree and did not sufficiently resemble the given function. Some justification for this feeling can be derived from the following argument.

We will require the concept of a fundamental Chebychev set.

Definition 4.21: The set $\{\varphi_k\}$ of the Chebychev approximating class L_n is said to form a fundamental Chebychev set if each element of $C[0, 1]$ can be arbitrarily well approximated by linear combinations of elements of the set $\{\varphi_k\}$ (Cheney, 1966, p.87).

Theorem 4.22: If the set of functions $\{\varphi_k\}$ of the Chebychev approximating class L_n forms a fundamental Chebychev set, then $\alpha^* \in R$, for n sufficiently large.

Proof: Let us fix the degree of the Chebychev class L_n. If the B.R.A. has a straddle point, then for that rotation of the function, any approximation of higher degree will not be any better approximation. But since a fundamental Chebychev set can uniformly approximate any continuous function, then if the degree is sufficiently large, $\alpha^* \in R$.

CHAPTER V

BEST ROTATED APPROXIMATION: COMPUTATION

5.1 ALGORITHM FOR COMPUTING B.R.A.

In this chapter we shall be concerned with polynomial approximation and hence with the error function (2.4) which has n linear and one non-linear parameter. The technique we describe for computing α^* and the B.R.A. is an iterated linear programming approach due to Esch and Eastman (1968) which has the advantage of not depending on characterization properties of equioscillating extrema for minimax approximation.

We wish to minimize h subject to the constraints

$$|h| \geq |e(t_j, \lambda)| \qquad j = 1, \ldots, N$$

which can be rewritten

$$h - e(t_j, \lambda) \geq 0 \qquad j = 1, \ldots, N$$

$$h + e(t_j, \lambda) \geq 0$$

and is equivalent to

$$\left.\begin{aligned} h - y_\alpha(t_j) + \sum_{k=0}^{n-1} a_k x_\alpha^k(t_j) &\geq 0 \\ h + y_\alpha(t_j) - \sum_{k=0}^{n-1} a_k x_\alpha^k(t_j) &\geq 0 \end{aligned}\right\} \quad j = 1, \ldots, N. \tag{5.1}$$

Equation (5.1) is not in linear programming form since the parameter α enters non-linearly. Our goal is to linearize (5.1) in α so that we may use linear programming methods for its solution.

Let $\alpha = \alpha_o + \delta\alpha$. Then

$$y_\alpha(t) = -x_{\alpha_o}(t)\sin\delta\alpha + y_{\alpha_o}(t)\cos\delta\alpha$$

$$\doteq -\delta\alpha x_{\alpha_o}(t) + y_{\alpha_o}(t)$$

$$x_\alpha(t) = x_{\alpha_o}(t)\cos\delta\alpha + y_{\alpha_o}(t)\sin\delta\alpha \qquad (5.2)$$

$$\doteq x_{\alpha_o}(t) + \delta\alpha y_{\alpha_o}(t)$$

$$x_\alpha^{n-1}(t) = (x_{\alpha_o}(t)\cos\delta\alpha + y_{\alpha_o}(t)\sin\delta\alpha)^{n-1}$$

$$x_\alpha^{n-1}(t) \doteq x_{\alpha_o}^{n-1}(t) + \delta\alpha(n-1)x_{\alpha_o}^{n-2}(t)y_{\alpha_o}(t).$$

Using (5.2) the system (5.1) can be linearized in the following way where for convenience we shall write only the second term in (5.1):

$$h + (1-a_1\delta\alpha)y_{\alpha_o}(t_j) - a_o - (a_1+\delta\alpha)x_{\alpha_o}(t_j)$$

$$-a_2x_{\alpha_o}^2(t_j) - 2a_2\delta\alpha x_{\alpha_o}(t_j)y_{\alpha_o}(t_j)$$

$$-a_3x_{\alpha_o}^3(t_j) - 3a_3\delta\alpha x_{\alpha_o}^2(t_j)y_{\alpha_o}(t_j) \quad j = 1,\dots,N \qquad (5.3)$$

$$-\dots - a_{n-1}x_{\alpha_o}^{n-1}(t_j) - (n-1)a_{n-1}\delta\alpha x_{\alpha_o}^{n-2}(t_j)y_{\alpha_o}(t_j) \geq 0.$$

Finally, we rewrite (5.3) as

$$y_{\alpha_o}(t_j) + h' \quad -A_o - A_1x_{\alpha_o}(t_j) - A_2x_{\alpha_o}^2(t_j) - B_2x_{\alpha_o}(t_j)y_{\alpha_o}(t_j)$$

$$-A_3x_{\alpha_o}^3(t_j) - B_3x_{\alpha_o}^2(t_j)y_{\alpha_o}(t_j) - \dots \quad j = 1,\dots,N$$

$$-A_{n-1}x_{\alpha_o}^{n-1}(t_j) - B_{n-1}x_{\alpha_o}^{n-2}(t_j)y_{\alpha_o}(t_j) \geq 0. \qquad (5.4)$$

On a given iteration, we start with a guess α_o and obtain as a solution of the linear program the values $A_o, \dots, A_{n-1}, B_2, \dots, B_{n-1}$ which gives the best approximation of (5.4) over the discrete point set

$\{t_j\}$. We can calculate

$$\delta\alpha = B_2/2A_2$$

and replace α_o by the hopefully better estimate $\alpha_o + \delta\alpha$. In order to limit $\delta\alpha$ so as to prevent the process from moving too far from the region in which the linearization is accurate, constraints of the form

$$\begin{aligned} 2\epsilon\,|A_{2(old)}| \pm B_2 &\geq 0 \\ &\vdots \\ (n-1)\epsilon\,|A_{n-1(old)}| \pm B_{n-1} &\geq 0 \end{aligned} \tag{5.5}$$

are incorporated into the program. The iterative process is halted when $\delta\alpha$ becomes less than some preassigned tolerance. At a solution, $A_i \doteq a_i$ and $h^* \doteq h'$.

5.2 COMPUTATIONAL EXPERIENCE

When α is given, $e^*(\alpha)$ is easily computed by standard linear programming techniques. Since the parameter space of α in the α-rotation error $e_\alpha(t)$ is $[0, \pi]$, a parameter search in α for the B.R.A. is feasible. With only a few exceptions (functions in Table 5.2 not appearing in Table 5.1), the results of the following section were computed in this manner. Although a parameter search provides the only real assurance that a B.R.A. has been found, it is something less than a practical solution. The algorithm of section 5.1 was implemented in the case of quadratic approximation. The linearization tolerance was set at .1, the iterative tolerance at 10^{-5}, α_o was taken at 0, and $A_{2(old)}$ at 1 for the initial iteration. The algorithm proved to be very efficient and accurate requiring no more than thirteen iterations, if it converged at all. Normally, for the last three iterations, the linear constraint was inactive. However, the algorithm did not converge for the functions: $x^2e^{-x^2}$, e^{-x^2}, x^2e^{-x}. For all three cases the B.R.A. has four critical points at α^*. For all other cases, the B.R.A. had five critical points at α^*. It is also true that for the functions $x^2e^{-x^2}$, e^{-x^2}, x^2e^{-x}, $e^*(\alpha)$ has a very flat slope for a large neighborhood of α^* (c.f. Fig. 5.1). At present the question is open whether the non-convergence of the algorithm is due to roundoff error or to theoretical reasons associated with the fact that the B.R.A. has only the necessary number of critical points at α^*. A similar phenomenon was observed for the function e^{2x}. For the function e^{2x}, $e^*(\alpha)$ has two relative mins, one at $\alpha = .082$, with five critical points and another at $\alpha = 1.13$ with four critical points. Regardless of the starting value α_o or the linearization tolerances, the algorithm converged to $\alpha^* = .082$.

5.3 COMPUTED RESULTS

In this section we discuss some of the numerical results we have obtained for second and third degree polynomial approximation, using the algorithm of section 5.1 and a parameter search program, for computing the optimal orientation α^*. These results are summarized in Tables 5.1 - 5.2. For all functions listed, $x \in [0, 1]$, the discretized point set consists of 101 points evenly distributed over the interval. In Table 5.1, for each function, the first entry under minimax error, for quadratic and cubic approximation, is the minimax error for $\alpha = 0$. The second row gives the minimax error at a relative min, the angle at which the relative min is achieved, and the number of critical points of the error function at the relative min. Often two relative mins were found. In these cases, the minimax error, the angle, and the number of critical points are again tabulated. In Table 5.2, we compare the unrotated minimax quadratic error with the best rotated quadratic minimax error and the unrotated minimax cubic error. The minimax errors of the third and fourth column both have the same number of effective parameters. In Fig. 5.1, the minimax error $e^*(\alpha)$ is graphed for the function $x^2 e^{-x^2}$, $\alpha \in [-\pi/2, \pi/2]$, for quadratic approximation. Figures 5.2 - 5.5 are graphs of $e^*(\alpha)$ for the function e^{2x}. Fig. 5.2 is for $\alpha \in [-\pi/2\ \pi/2]$, quadratic approximation; Fig. 5.3 for $\alpha \in [0, \pi/2]$, quadratic approximation; Fig. 5.4 for $\alpha \in [-\pi/2, \pi/2]$, cubic approximation; and Fig. 5.5 for $\alpha \in [0, \pi/2]$, cubic approximation. Fig. 5.6 is a plot of the function e^{2x} and its minimax approximation together with the rotation of e^{2x} by α^* and its approximation by the B.R.A.

5.4 DISCUSSION

From Table 5.2 it is evident that the minimax error at a B.R.A. may or may not be smaller than the error at a cubic minimax approximation. Two extreme examples of functions exhibiting this behavior are the following: If $f(x) = x^3$, the quadratic B.R.A. is much worse than the cubic approximation. If $f(x) = \sqrt{x}$, the cubic approximation, or any polynomial approximation, is much worse than the quadratic B.R.A. However we note that a second degree B.R.A. which has comparable error to a cubic approximation is a better curve fit since it has fewer wiggles. For the examples given, the B.R.A. often has significantly smaller error than the unrotated minimax approximation of same degree. It also seems significant that α^* is often small so that even a slight change in the orientation of the data with respect to rotation can dramatically effect the curve fit. We note that the ratio of quadratic B.R.A. error to second degree minimax approximation is a factor of twenty for the function e^{2x} and a factor of fifty-eight for the function e^{x^2}. Typical improvement seems to be in the order of factors of five to ten. Fig. 5.6 demonstrates the effect of a twenty times better curve fit to the function e^{2x}. In this case, the B.R.A. is essentially indistinguishable from the curve.

An examination of the data in Table 5.1 reveals that for most of the cases examined, the B.R.A. error had n+2 equioscillating extrema, rather than the necessary n+1. In earlier stages of this study, it was anticipated that n+2 equioscillating extrema characterized, in some way, a B.R.A. However, this is not the case. The example $x^2e^{-x^2}$, quadratic approximation, demonstrates that n+2 equioscillating extrema is not necessary at a B.R.A. The function e^{2x}, cubic approximation, demonstrates that the condition is not sufficient. The function e^{2x}, quadratic approximation, had a relative min with n+1 equioscillating extrema, demonstrating that even for convex functions, n+2 equioscillating extrema is not necessary at a relative min. For the function $x^2e^{-x^2}$, cubic approximation, n+2 equioscillating extrema were observed at $\alpha = 1.306$. At this point, $e^*(\alpha)$ was not at a relative min. Hence n+2 equioscillating extrema is not locally sufficient.

The function $\sqrt{x}, x \in [0, 1]$, has been useful as a counter-example to many conjectures concerning the characterization of α^*. It may be conjectured that the angle of rotation for which the range of the derivative in absolute value is a minimum should be α^*. However, this is not true for $\sqrt{x}$. It may also be conjectured that the angle for which the modulus of continuity is least should be α^*. This is also contradicted by the $\sqrt{x}$ function.

5.5 CONVEX FUNCTIONS OF nth DEGREE

It is easy to see that if $f''(x) > 0$, $x \in (0,1)$, then the B.A. straight line must have critical points at the end of the interval. A general result of this type is also true; i.e., if $f^{(n)}(x) > 0$, $x \in (0,1)$, then the B.A. polynomial of degree n-1 has critical points at the end of the interval. It is also true that if $f^{(n)}(x) > 0$, $x \in (0,1)$, then there must exist exactly n+1 critical points at a B.A. Let us denote the nth derivative of an α-rotation of f at t in the x, y coordinate system as $f_\alpha^{(n)}(t)$. From the above discussion it follows that if a B.R.A. has n+2 critical points when $\alpha^* \in R$, then $f_{\alpha^*}^{(n)}(x) \not> 0$, $x \in (0,1)$.

The first four derivatives of an α-rotated curve are as follows:

$$f'_\alpha(t) = y'_\alpha(t)/x'_\alpha(t) = \frac{-\sin\alpha + f'(t)\cos\alpha}{x'_\alpha(t)} \tag{5.6}$$

$$f''_\alpha(t) = \frac{f''(t)}{(x'_\alpha(t))^2}$$

$$f_\alpha^{(3)}(t) = \frac{f'''(t)\cos\alpha + (f'''(t)f'(t) - 2(f''(t))^2)\sin\alpha}{(x'_\alpha(t))^3}$$

$$f_\alpha^{(4)}(t) = \frac{\begin{array}{c} f^{(4)}(t)\cos^2\alpha + (f^{(4)}(t)f'(t) - 6f'''(t)f''(t))\sin\alpha\cos\alpha \\ + (f^{(4)}(t)(f'(t))^2 - 6f'(t)f''(t)f'''(t) - 6(f''(t))^3)\sin^2\alpha \end{array}}{(x'_\alpha(t))^4}$$

The denominator $x'_\alpha(t)$ is always positive or negative for α belonging to the interior of R. The numerator of $f_\alpha^{(3)}(t)$, for $f(x) = x^3$ is: $6\cos\alpha - 54x^2\sin\alpha$, which clearly changes sign for appropriate α, $x \in (0,1)$.

In a sense, the fact that the nth derivative of a function is a constant sign inhibits its "approximability" by limiting the number of extrema of the error function and hence the number of times the approximation can wind around the function. By introducing the rotation parameter α and searching for a B. R. A. we are, in effect, attempting to remove the limitation on the number of possible extrema of the error function.

TABLE 5.1

Function $x \in [0, 1]$	Quadratic approximation minimax error	α^*	# c.p.	Cubic approximation minimax error	α^*	# c.p.
x^4	.06345			.00781		
	.01114	.4130	5	.00169	.0647	6
				.00648	.8679	6
x^3	.03125			0		
	.00538	.2782	5			
sin2x	.02192			.00418		
	.00823	.1279	5	.00533	-.268	6
				.000321	.088	6
e^x	.000875			.000543		
	.000250	.2864	5	.0000146	.0817	6
				.0000251	.62	6
e^{2x}	.12237			.01501		
	.00645	.0818	5	.000871	.0229	6
	.03441	1.13	4	.00138	.1957	6
xe^{-x}	.00818			.000706		
	.000659	.6144	5	.0000414	.08	6
				.000187	.970	6
$x^2e^{-x^2}$	.02607			.00201		
	.02405	.40	4	.00109	.34	5
x^2e^{-x}	.01128			.00170		
	.01092	.25	4	.000974	-.588	6
e^{-x^2}	.01787			.000658		
	.01753	-.216	4	.000377	-.044	6
xe^{-x^2}	.00617			.00452		
	.00481	-.0750	5	.000882	.448	6
				.00285	-.433	6
$e^x/(x+1)$	.00123			.000545		
	.000546	-.1484	5	.000537	-.076	5
$e^{x^2}/(x+1)$	.01280			.00664		
	.00584	.0873	5	.00584	.09	5

TABLE 5.2

Function $x \in [0, 1]$	Quadratic minimax error	Quadratic B.R.A. error	Cubic minimax error	α^*
x^3	.03125	.00538	.0	.2782
x^4	.06345	.01114	.00781	.4130
x^5	.09216	.01647	.01971	.4922
x^6	.11704	.02121	.03326	.5438
e^x	.000875	.000250	.000543	.2864
e^{2x}	.12237	.00645	.01501	.0818
e^{3x}	.74832	.05547	.13494	.0288
e^{4x}	3.32688	.30375	.77822	.0108
e^{-x}	.00322	.000111	.000198	-.6757
e^{-2x}	.01656	.00102	.00203	-.5445
e^{-3x}	.03726	.00319	.00672	-.5246
e^{x^2}	.05894	.00154	.01268	.2223
xe^{-x}	.00818	.000659	.000706	.6144
$x^2e^{-x^2}$	.02607	.02405	.00201	.40
x^2e^{-x}	.01128	.01092	.00170	.25
e^{-x^2}	.01787	.01753	.000658	-.216
$\sin 2x$	.02192	.00823	.00418	-.1279
$e^x/(x+1)$	.00123	.000546	.000545	-.1484
$e^{x^2}/(x+1)$	.01280	.00584	.00664	.0873
xe^{-x^2}	.00617	.00481	.00452	-.0750

$f(x) = x^2 e^{-x^2}$

Quadratic approximation

$h_o^* = .02607$

$h^* = .02405$

$\alpha^* = .40$

four critical points

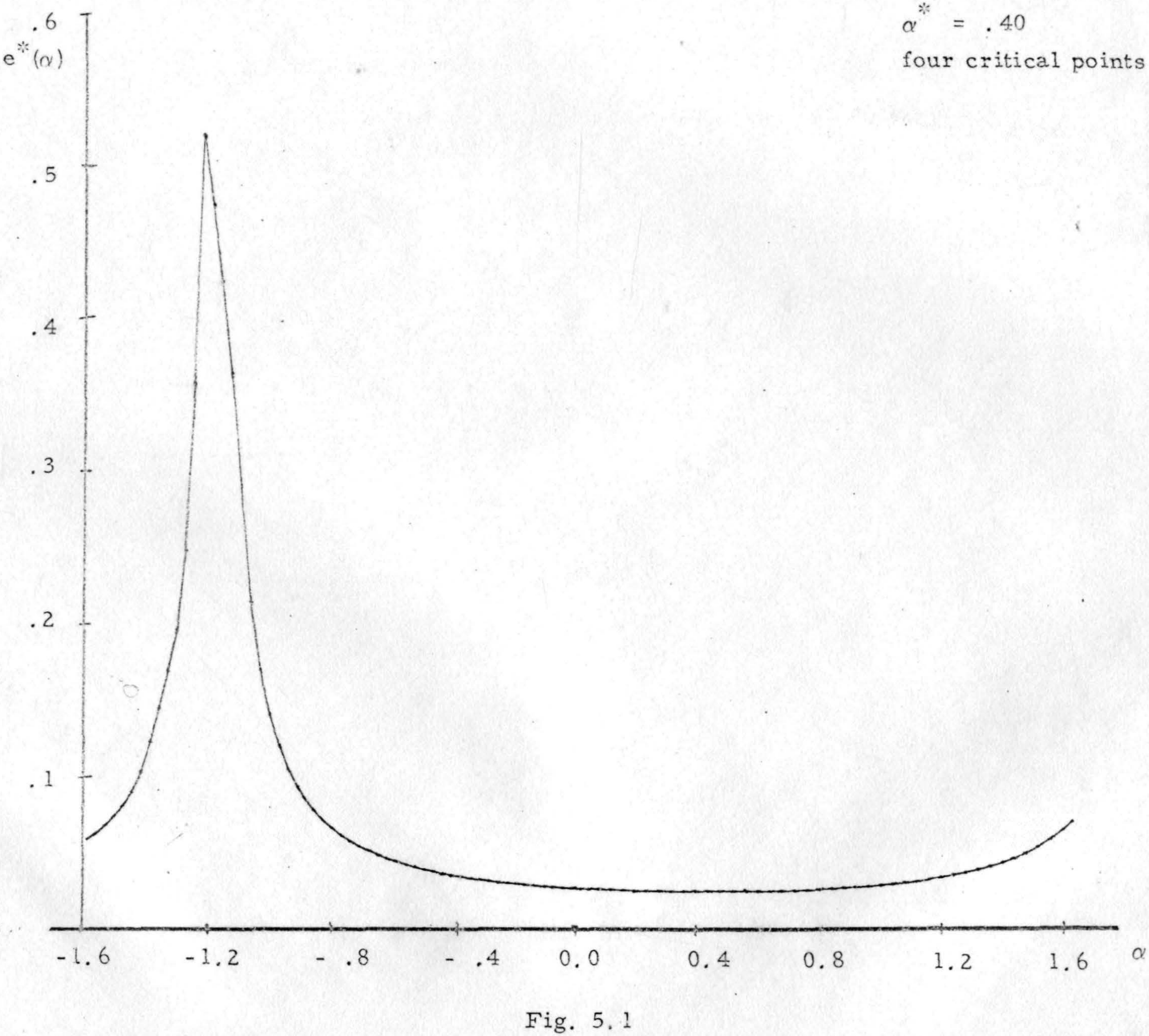


Fig. 5.1

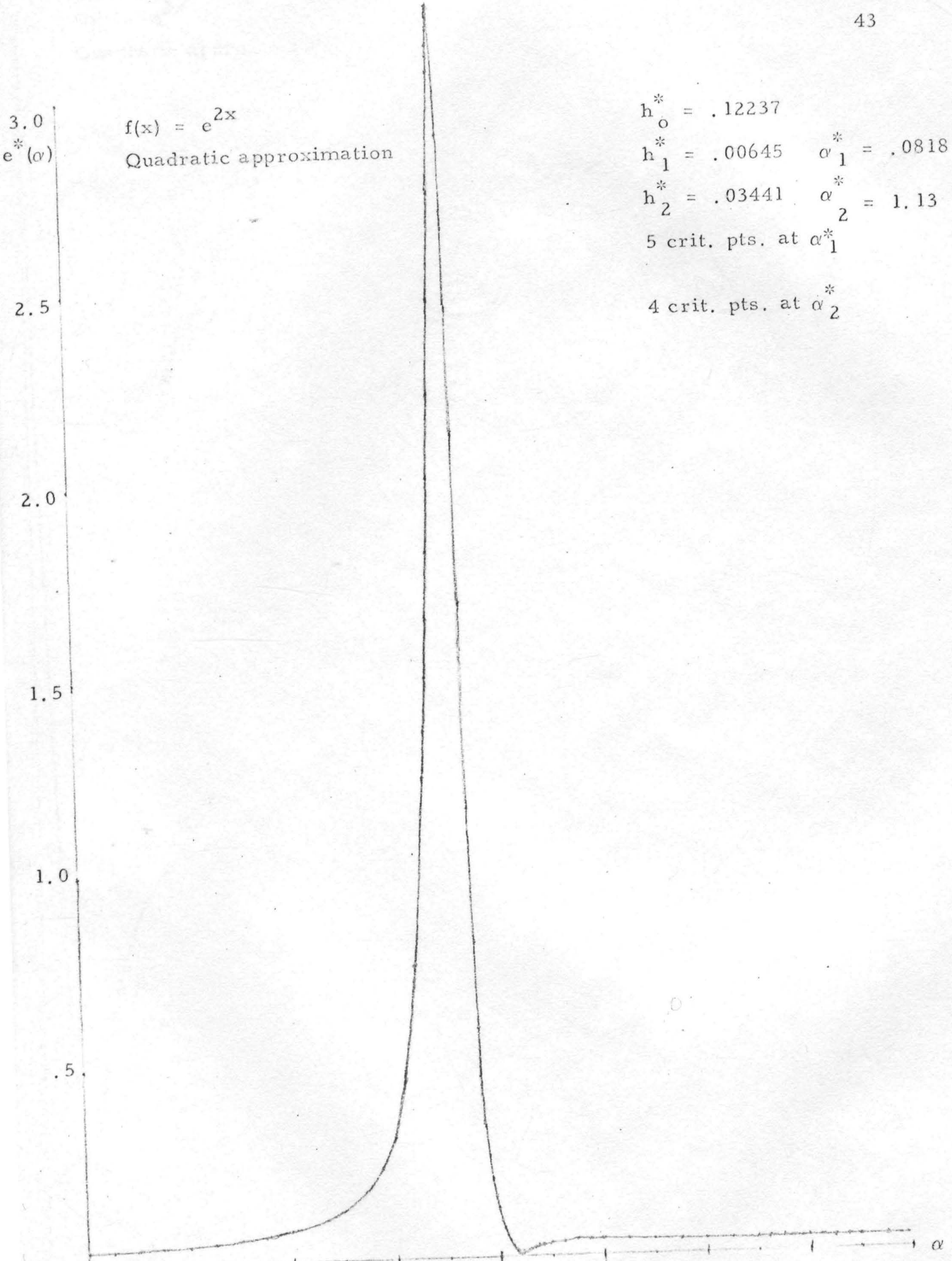

$f(x) = e^{2x}$
Quadratic approximation
$h_o^* = .12237$
$h_1^* = .00645$ $\alpha_1^* = .0818$
$h_2^* = .03441$ $\alpha_2^* = 1.13$
5 crit. pts. at α_1^*
4 crit. pts. at α_2^*
3.0
$e^*(\alpha)$
2.5
2.0
1.5
1.0
.5
-1.6
-0.8
0.0
0.8
1.6
α

Fig. 5.2

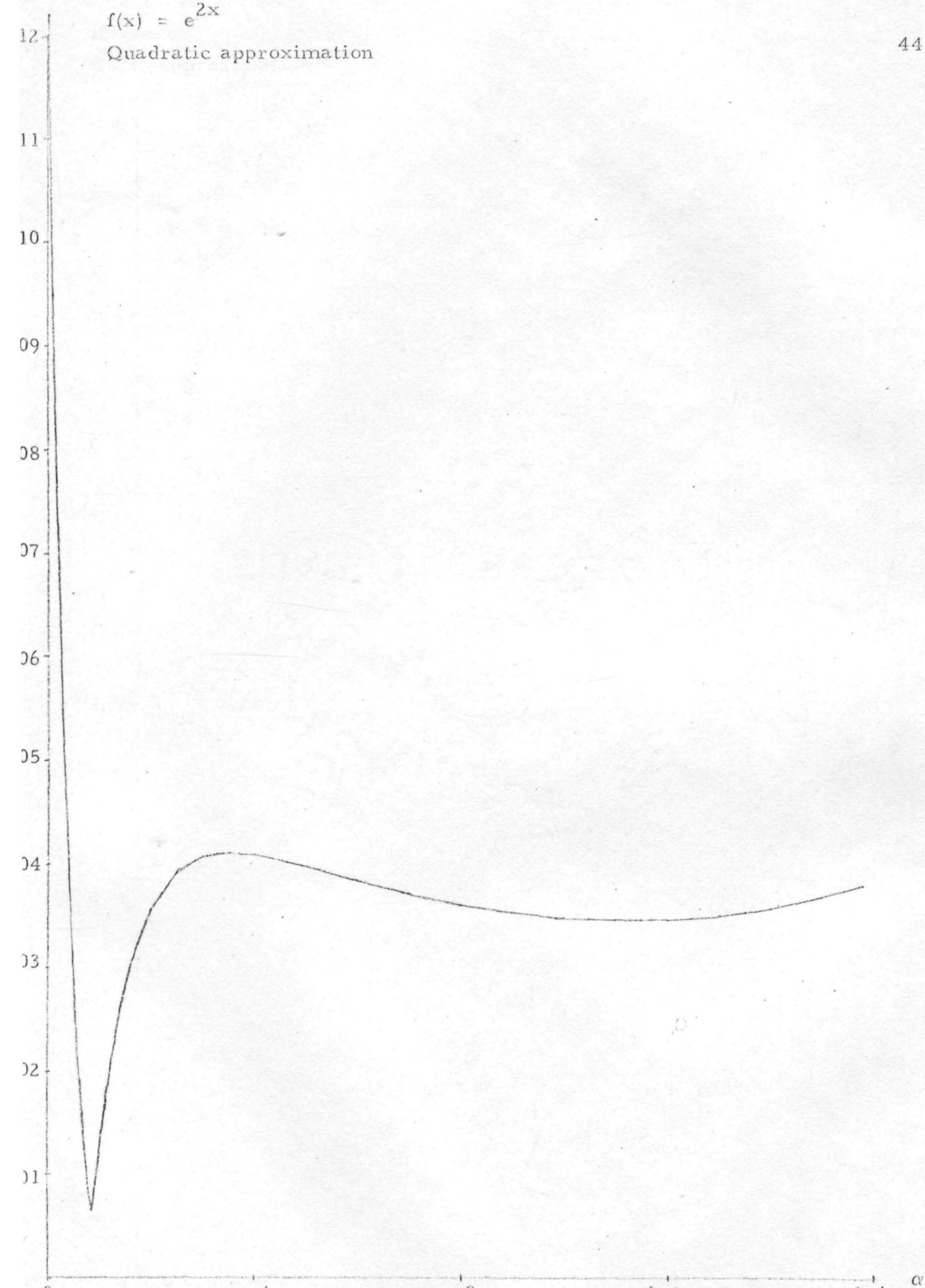
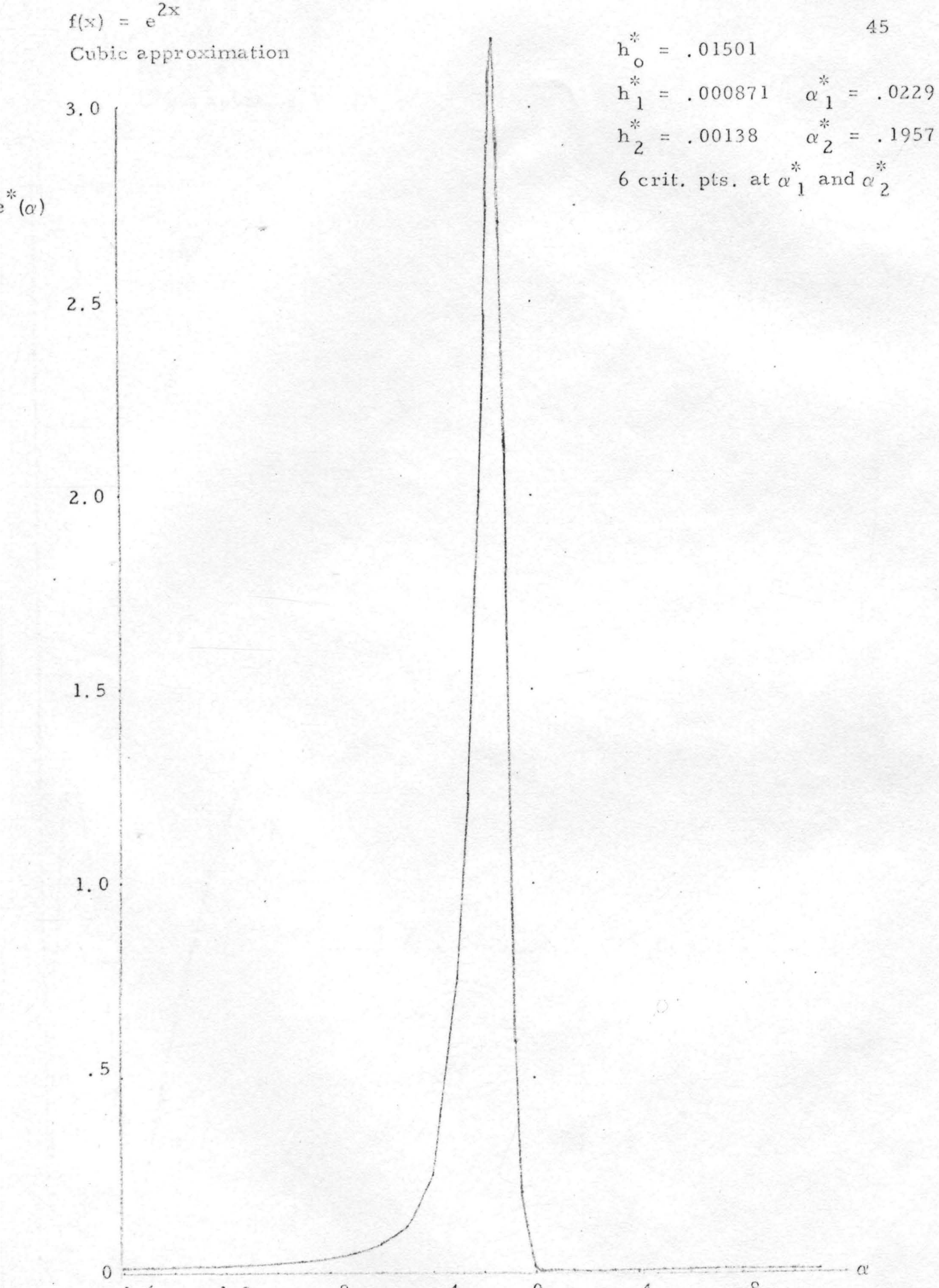

f(x) = e^2x
Quadratic approximation
12
11
10
09
08
07
06
05
04
03
02
01
0
.4
.8
1.2
1.6
α

Fig. 5.3



f(x) = e^2x
Cubic approximation
h*o = .01501
h*1 = .000871 α*1 = .0229
h*2 = .00138 α*2 = .1957
6 crit. pts. at α*1 and α*2
e*(α)
3.0
2.5
2.0
1.5
1.0
.5
0
-1.6
-1.2
-.8
-.4
0
.4
.8
α

Fig. 5.4

$f(x) = e^{2x}$

Cubic approximation

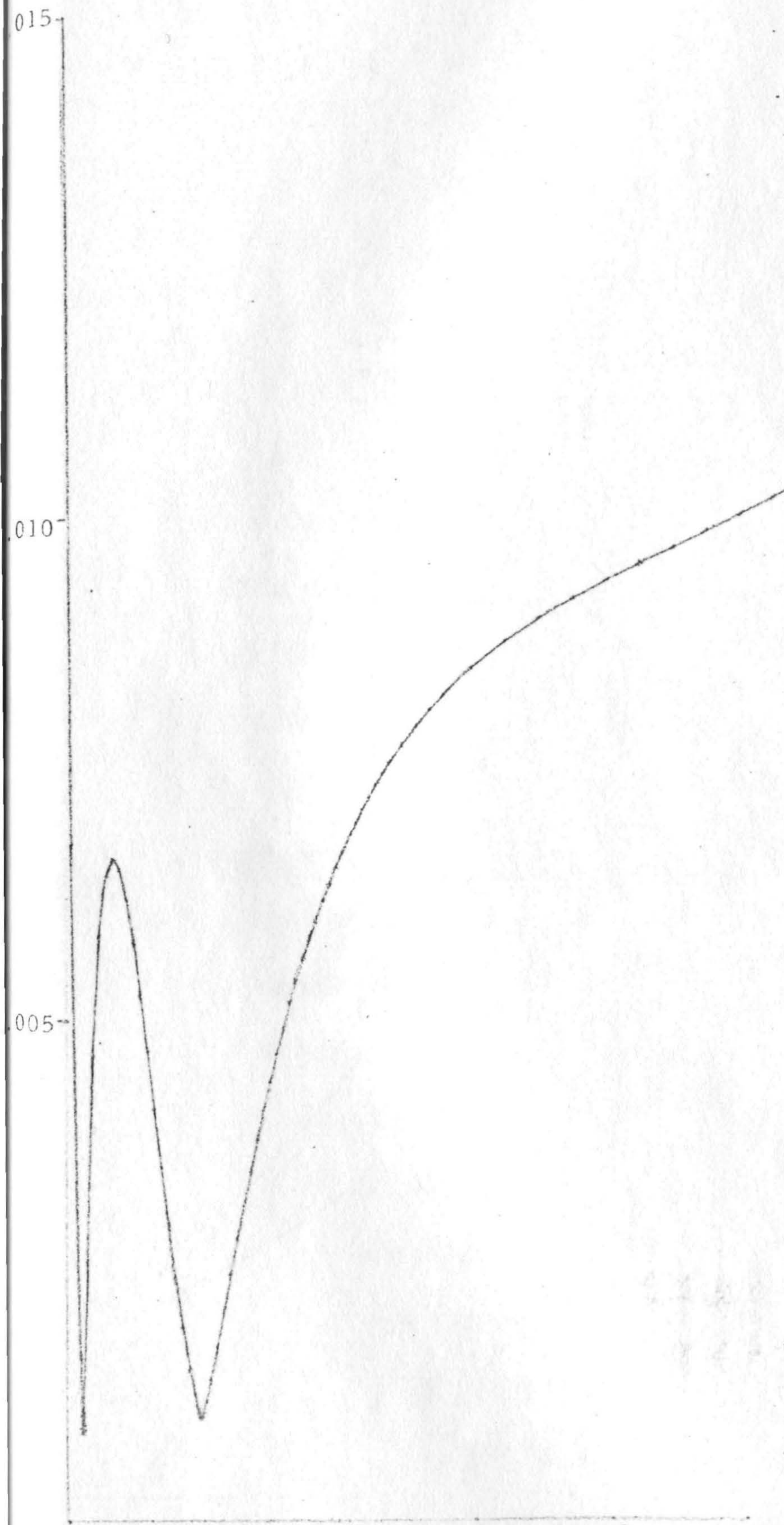


Fig. 5.5

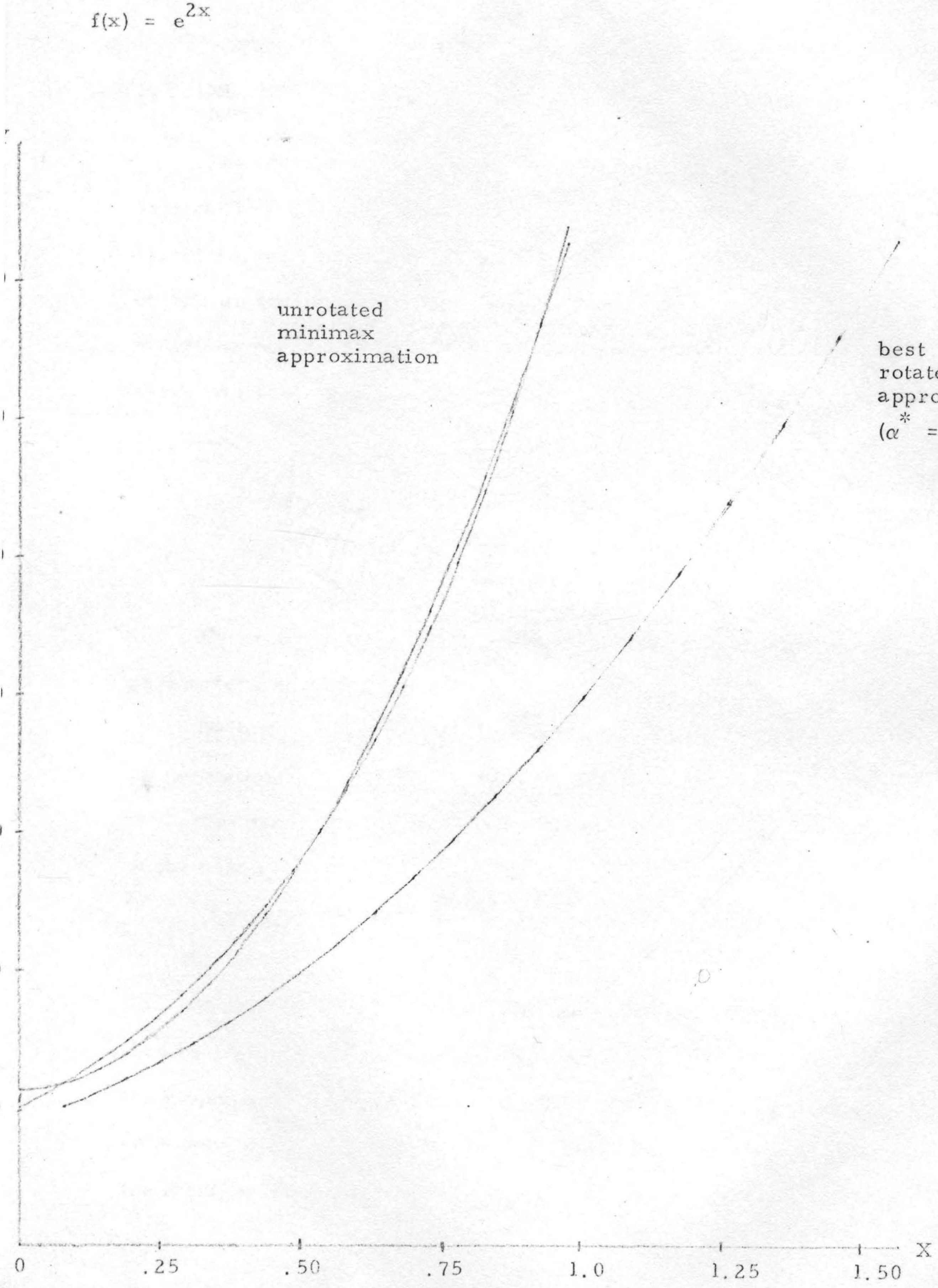

f(x) = e^2x
unrotated
minimax
approximation
best
rotated
approximation
(α* = 4.7°)
0
.25
.50
.75
1.0
1.25
1.50
X

Fig. 5.6

CHAPTER VI

BEST ROTATED STRAIGHT LINE APPROXIMATIONS

6.1 NECESSARY CONDITIONS FOR BEST ROTATED STRAIGHT LINE APPROXIMATIONS

The principal simplification which results from considering straight line approximations is that the α-rotation of a straight line is a straight line. Hence a best approximating straight line for an α-rotation of f is an admissible approximation of an α'-rotation of f. The following equations relate the slope and intercept of an α-rotated straight line $cx+d$ to the slope and intercept of the same straight line without rotation:

$$\begin{aligned} c &= (a\cos\alpha - \sin\alpha)/(a\sin\alpha + \cos\alpha) \\ d &= b/(a\sin\alpha + \cos\alpha) \\ a &= (\sin\alpha + c\cos\alpha)/(\cos\alpha - c\sin\alpha) \\ b &= d/(\cos\alpha - c\sin\alpha)\ . \end{aligned} \tag{6.1}$$

Hence we can describe the α-rotation of any straight line with the parameters a, b and α.

Definition 6.1: $F(a,b,\alpha)$ is the uniform error of approximation of an α-rotation of the straight line $ax+b$ for an α-rotation of f. $F(a^*,b^*,\alpha)$ is the minimax error for an α-rotation of f and $F(a^*,b^*,\alpha^*)$ is the error of the best rotated approximation.

Theorem 6.2: $F(a^*,b^*,\alpha^*) = F(\tan\alpha^*, b^*, \alpha^*)$; i.e., $c^* = 0$ at a B.R.A.

Proof: If the slope of the best approximating line is not zero, then by rigidly rotating f and the approximating line an angle α' so that the slope of the straight line is now zero, the error of approximation at each point of the curve is the perpendicular distance from the curve to the straight line in the original α-rotation. Hence the error of approximation at α' is less than at α. If $c^* = 0$, $a^* = \tan\alpha^*$.

Corollary 6.3: If straight line B.A. is non-unique at α_o, then $\alpha_o \neq \alpha^*$.

Proof: If B.A. at α_o is non-unique, then there exists a straddle point extremum and hence a B.A. such that $c^* \neq 0$.

Definition 6.4: $G(k, \beta)$ is the uniform error of a constant approximation for a β-rotation of f. $G(k^*, \beta^*) \leq G(k, \beta)$ for all k, β.

Corollary 6.5: $F(a^*, b^*, \alpha^*) = G(k^*, \beta^*)$, and $\alpha^* = \beta^*$, $k^* = d^*$.

Theorem 6.6: A necessary condition that a constant approximation be a B.R.A. is that the error function $e_\alpha(t)$ has three extrema with characterization according to theorem 3.13 or a straddle point in $x_\alpha(t)$.

Theorem 6.7: A necessary condition for a best rotated straight line approximation is that the straight line is a B.A. and that $a^* = \tan\alpha^*$.

An application of theorem 6.2 is the following: Let $f(x) = x^n$, (Fig. 6.1) $x \in [0, 1]$, $n \geq 2$. Under the assumption that $\alpha^* \in R$, the best approximating straight line must be a constant with three alternation points. Therefore it follows that the orientation of $y_{\alpha^*}(t)$ is Fig. 6.2.

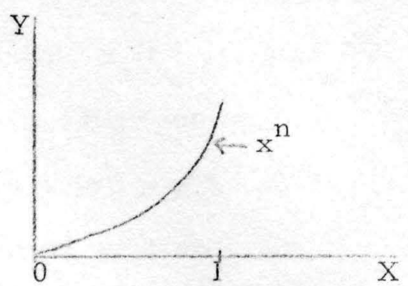


Fig. 6.1

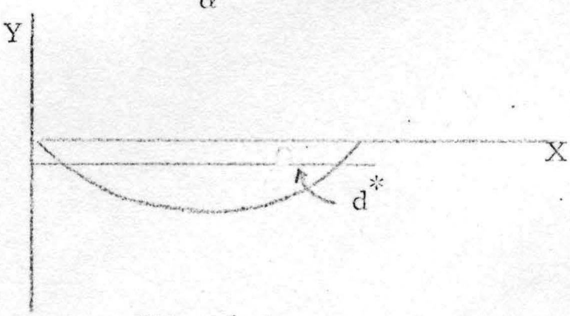


Fig. 6.2

Thus for all n, $\alpha^* = 45^\circ$, $a^* = 1$, $d^* = -\epsilon$, and from (6.1) $b^* = -\sqrt{2}\,\epsilon$. Here ϵ depends on n, and $\epsilon \to \sqrt{2}/4$ as $n \to \infty$.

6.2 ROTATION INVARIANCE OF BEST APPROXIMATING STRAIGHT LINES

For straight line approximation we can ask whether there exists any relation between best approximations for different rotations of f.

Definition 6.8: Let $a^*x + b^*$ be a B.A. for a fixed rotation of f. If for some range T of α,

$$F(a^*, b^*, \alpha), \ \alpha \in T$$

is the error of best approximation, then we say that $a^*x + b^*$ is rotation invariant for $\alpha \in T$.

Theorem 6.9: If f is continuous on $[0,1]$, $G_2 = P_1$, then the B.A. to f, $a^*x + b^*$, is rotation invariant for $\alpha \in R$.

Proof: For straight line approximation, the error function is

$$e_\alpha(t) = y_\alpha(t) - cx_\alpha(t) - d. \qquad (6.2)$$

From equation (6.1), we can put (6.2) in the form

$$e_\alpha(t) = \frac{f(t)-at-b}{a \sin\alpha + \cos\alpha} . \qquad (6.3)$$

Let t_1, t_2, t_3 be the critical point set belonging to $[0,1]$ for the B.A. a^*t+b^* to f (without rotation). By definition $t_1 < t_2 < t_3$, and for $\alpha \in R$, $x_\alpha(t_i)$ forms a critical point set for (6.2).

It is interesting to note that the proof of theorem 6.9 will not work for $\alpha \notin R$, since $x_\alpha(t_i)$ is not necessarily a critical point set, for these rotations.

Theorem 6.10: If $\alpha \notin R$, straight line approximations for continuous functions are not necessarily rotation invariant.

Proof: From Fig. 4.1 and 2.2, the result follows.

The result, theorem 6.9, makes it a particularly simple matter to compute the best rotated straight line approximation if we know that $\alpha^* \in R$. Given the B.A. for f without rotation, $\alpha^* = \tan^{-1} a^*$.

CHAPTER VII

EXTENSION OF SOME RESULTS OF TORNHEIM

Tornheim (1950) gave the following definitions:

Definition 7.1: A class of functions C_n is said to be convex with respect to a unisolvent class G_n of degree n on $[a, b]$, if $f \in C_n$ is continuous on $[a, b]$ and has at most n intersections with any member $g \in G_n$.

Definition 7.2: A graze point is a point of intersection of $g(x)$ with $f(x)$ such that $f(x)-g(x)$ does not change sign in a suitably small neighborhood of the intersection.

Tornheim then proved that if $f \in C_n$ on $[a, b]$, and if $g \in G_n$, a unisolvent class on $[a, b]$, then $f(x)-g(x)$ has no graze points, under the condition that f has n intersections with g or $n-1$ intersections with g such that the intersections do not occur at a or b and such that $\text{sign}(f(a)-g(a)) = (-1)^{n+1} \text{sign}(f(b)-g(b))$ is verified.

These results directly relate to the form of the error function at a minimax approximation since $g^*(x)$ necessarily has n intersections with f. Therefore, if $f \in C_n$, $e^*(x)$ has no graze points.

The computer results of chapter V has motivated us to consider the form of the error function at a best rotated approximation. In this case, f must be allowed $n+1$ intersections with members of G_n.

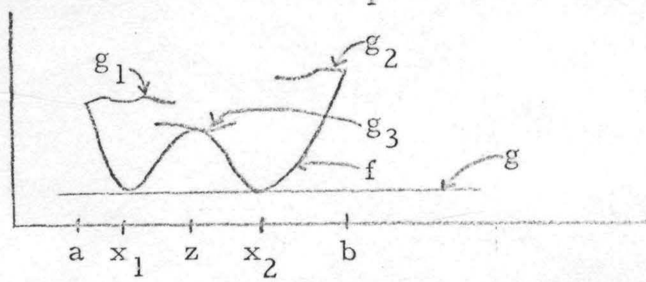
Definition 7.3: $f \in C_{n+1}$ with respect to a unisolvent class G_n of degree n on $[a, b]$, if f is continuous on $[a, b]$ and f has at most $n+1$ intersections with any member $g \in G_n$, $x \in [a, b]$.

In the theorems which follow we will show that the results of Tornheim do generalize to C_{n+1} classes of functions. We also note that both theorems will be of use in characterizing the form of the error function at a B. R. A., where, it is assumed, $\alpha^* \in R$, and $f \in C_{n+1}$.

Theorem 7.4: If $f \in C_{n+1}$ and has n+1 intersections with $g \in G_n$, a unisolvent class of degree n on [a,b], then f does not graze g anywhere, $x \in (a,b)$.

Proof: Following Tornheim, we shall prove the theorem for n=1 and then reduce the general case to this one. Let x_1 and x_2 be the two points of intersection, belonging to (a,b). Case A: x_1 and x_2 are both graze points. Case B: Only x_1 is a graze point.

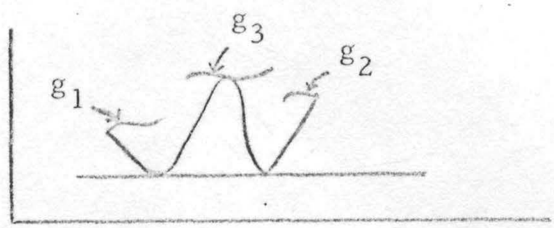
Case A:



We define $g_1, g_2, g_3 \in G_n$ such that $g_1(a) = f(a)$, $g_2(b) = f(b)$ and $g_3(z) = f(z)$, where $x_1 < z < x_2$. Since f-g does not change sign on [a,b], we will assume $f-g \geq 0$, and is therefore zero only at x_1 and x_2. Hence $g_i > g$, for all g_i. We will not consider the cases $g_1 > g_2$ and $g_2 > g_1$ distinct and shall assume $g_2 > g_1$. We will assume that all the g_i's are distinct. Hence, there are three distinct cases which we must consider: 1) $g_3 > g_2 > g_1 > g$; 2) $g_2 > g_3 > g_1 > g$; 3) $g_2 > g_1 > g_3 > g$. We note that for a unisolvent class of degree one, if two members of the class intersect, they are equal over the whole interval.

If g_3 is not a graze point, we are immediately led to a contradiction. Hence we shall assume g_3 is at a graze point at z.

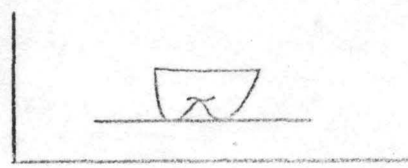
A1:



Case A1 is impossible since g_2 must be below g_3 and above g without intersecting either, which implies g_2 must intersect f more than once more. Cases A2 and A3 follow similarly.

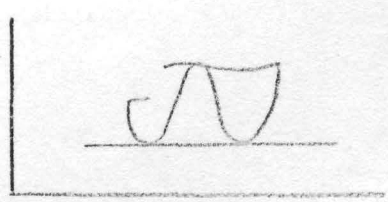
For cases A1, A2, and A3, the g_i's were assumed distinct.

A4: $g_1 = g_2$



Under the Case A4 assumptions, there can be no other intersection of g_1 with f. Hence $g_1 - f > 0$, $x \in (a, b)$. If $g_3 > g_1$, the contradiction is immediate. If $g_1 > g_3$, then g_3 is below g_1 and above g which implies it intersects f more than once more.

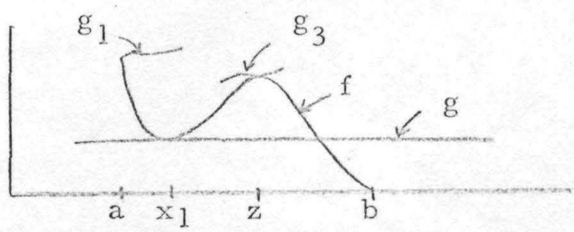
A5: $g_2 = g_3$ (same as $g_1 = g_3$)



Under the Case A5 assumptions, if $g_2 > g_1$, then g_1 must have more than one more intersection with f. If $g_1 > g_2$, then g_2 must have more than one more intersection with f.

Finally, $g_1 = g_2 = g_3$ is impossible since g_1 then has three intersections with f.

Case B:



The functions $g_1, g_3 \in G_1$ are assumed distinct and defined as in Case A. If $g_3 > g_1$, then g_1 must have more than two intersections with f since $g_1 > g$. Hence, we assume $g_1 > g_3$. Under this assumption g_3 must have an intersection with f to the left of z (and hence z is a graze point). Since g_3 and g are distinct, they cannot intersect anywhere on $[a, b]$ and in particular at a. We define $g_2 \in G_1$ such that $g(a) < g_3(a)$. But this implies g_2 has three or more intersections with f, and therefor a contradiction.

If $g_1 = g_3$, we construct g_2 in the previous manner and again reach a contradiction. This finishes the cases for $n = 1$.

We next consider the general case. Suppose f grazes g at x_1. Let $[a', b']$ be a closed interval with x_1 in its interior and containing abscissas of none of the other n points of intersection. If we take only those functions g of G_n which pass through the other $n-1$ points of intersection and restrict them to $[a', b']$, we obtain a unisolvent class of degree one, G', in which f restricted to $[a', b']$ is a C_2 function f'. But then f' cannot graze any function of G', hence f does not graze g at x_1.

Corollary 7.5: If $f \in C_{n+1}$ and has n intersections with a $g \in G_n$ but not at $x = a$ or b, and if $\text{sign}(f(a)-g(a)) = (-1)^n \text{sign}(f(b)-g(b))$ then g and f intersect exactly n times.

Proof: There could be at most one more intersection and if this occurred f-g would change sign at each intersection according to theorem 7.4 so that $\text{sign}(f(a)-g(a)) = (-1)^{n+1}\text{sign}(f(b)-g(b))$. This is a contradiction to the hypothesis of the corollary.

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